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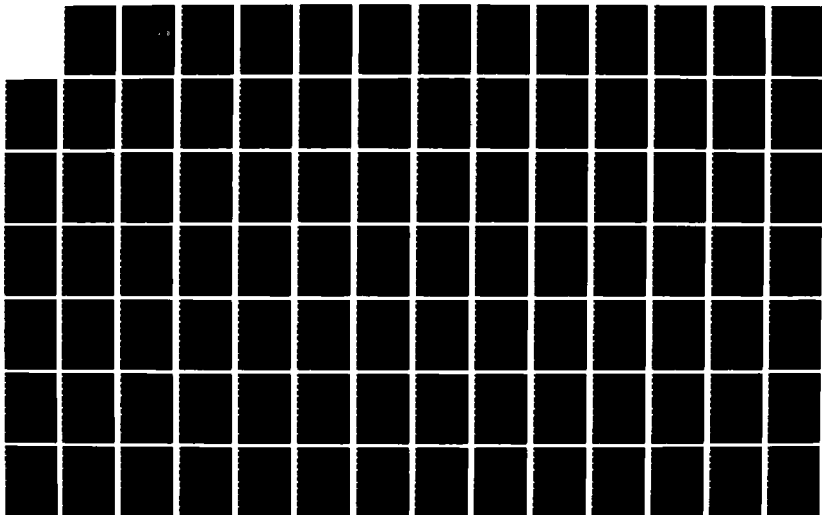
RESOLUTION OF A RANK-DEFICIENT ADJUSTMENT MODEL VIA AN  
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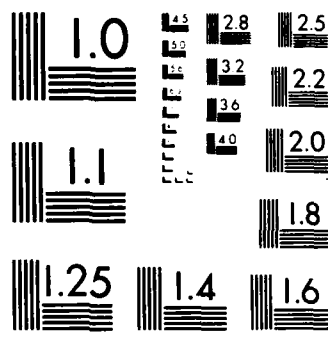
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**RESOLUTION OF A RANK-DEFICIENT ADJUSTMENT MODEL  
VIA AN ISOMORPHIC GEOMETRICAL SETUP  
WITH TENSOR STRUCTURE**

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Final Report  
21 February 1986 - 3 March 1987

March 1987


Approved for public release; distribution unlimited

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFGL-TR-87-0102	2. GOVT ACCESSION NO. ADA 184198	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) RESOLUTION OF A RANK DEFICIENT ADJUSTMENT MODEL VIA AN ISOMORPHIC GEOMETRICAL SETUP WITH TENSOR STRUCTURE		5. TYPE OF REPORT & PERIOD COVERED Final Report Period 2/21/86 3/3/87
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Georges Blaha		8. CONTRACT OR GRANT NUMBER(s) F19628-86-K-0006
9. PERFORMING ORGANIZATION NAME AND ADDRESS Nova University Oceanographic Center 8000 North Ocean Drive Dania, Florida 33004		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2309G1BL
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Geophysics Laboratory Hanscom AFB, Massachusetts 01731 Contract Monitor: Dr. Christopher Jekeli/LWG		12. REPORT DATE March 1987
		13. NUMBER OF PAGES 97
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report)  Approved for public release; distribution unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)		
Least squares method Variance covariances Weights Error propagation	Tensor analysis Orthonormal vectors Metric tensor Associated metric tensor	Covariant components Contravariant components Hypersurface Hyperplane
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)		
<p>This study develops the rank deficient adjustment theory in a geometrical manner. In accordance with most least-squares (L.S.) applications, the adjustment model is considered linear or linearized. The fundamental building blocks consist of orthonormal vectors spanning the spaces and surfaces linked to the L.S. setup. From this setup to the desired results including the variance-covariance matrices, the standard adjustment quantities can be represented by first- and second-order tensors. It is thus possible to express them in terms of the components of the above vectors, allowing for an easy and clearcut geometrical interpretation of the</p>		

L.S. process. By virtue of such a "vectorization", the propagation of the contravariant and covariant metric tensors is shown to fit perfectly the variance covariance propagation law and even to establish a weight propagation law. The opportunity to obtain variance-covariances and weights as a coherent part of the geometrical development provides the motivation for using tensor structure in the analysis of various L.S. methods and their properties.

In principle, an isomorphism between adjustments and geometry is rooted in the notion that a consistent model relationship restricts a general vector to a "model surface" (here a hyperplane). The mechanism is provided by the L.S. criterion, which projects an "observational vector"  $dx$  lying in an  $n$ -dimensional observational space orthogonally onto this  $u'$ -dimensional model surface. The projected vector is attributed the dual notation  $dx' \equiv du'$ . The observational space is spanned by  $n$  orthonormal vectors  $l, j, \dots, v, \dots$ , while the model surface embedded in it is spanned by  $u'$  orthonormal vectors  $l, j, \dots$ . In the rank-deficient problems, where the rank of the design matrix is  $u'$  and the rank deficit is  $u'' = u - u'$ , the model surface is also embedded in a  $u$ -dimensional parametric space spanned by  $u$  orthonormal vectors  $l, j, \dots, t, \dots$ , and is thus an intersection of the observational and the parametric spaces. The observational-space contravariant components of  $dx'$  represent the adjusted observations, and the parametric-space contravariant components of  $du'$  represent the adjusted parameters. These two kinds of components of the same geometrical object are related through the design matrix.

The isomorphic geometrical setup reveals that all the adjustment matrices, i.e., the design matrix, the variance-covariance matrices (of observations, adjusted observations, residuals, and parameters), and the corresponding weight matrices, can be expressed as a product of two constituent matrices each. This outcome is further qualified: (a) All constituent matrices are formed in terms of orthonormal vectors, the elementary geometrical objects; (b) These vectors are the same in either matrix of the constituent pair, only the type of their components may differ; and (c) The set  $l, j, \dots$  spanning the model surface is common to all constituent matrices except those pertaining to the residuals.

The geometrical development yields a general L.S. resolution, where the solution vector  $du'$  and its variance-covariance matrix  $a'$  are non-unique. This resolution is analyzed in three distinct formulations giving identical results. Two of these formulations utilize the matrix of minimal constraints, the first generating augmented observation equations and the second generating augmented normal equations. The third formulation analyzes the parametric-space components of the orthonormal vectors, showing that the properties of the resolution depend entirely on  $u'' \times u'$  free elements grouped in the matrix  $\Delta L''$ . A completely arbitrary  $\Delta L''$  produces the general resolution with non unique  $du'$  and  $a'$ . A specific data dependent restriction on  $\Delta L''$  yields the unique minimum norm solution  $du'$ , but a non unique  $a'$ . Finally, if  $\Delta L'' = 0$ , both  $du'$  and  $a'$  are unique. In this case  $du'$  is the minimum norm solution as above and  $a'$  is its variance-covariance matrix with the minimum trace. It can be concluded that the minimum trace criterion is superior to any other. Even if some of them produced unique  $du'$  and  $a'$ , the norm of  $du'$  would not be a minimum, or the trace of  $a'$  would not be a minimum, or both.

Other topics related to geometry with tensor structure are addressed as well. An algorithm furnishing the pseudoinverse of a positive semi definite matrix, which could be useful for its straightforward geometrical interpretation as well as for its computational efficiency, is developed as a by product of this analysis. The Choleski algorithm for positive definite as well as positive semi definite matrices is interpreted in terms of orthonormal vector components. Another item shows how the tensor structure developed herein could be useful in applications unrelated to adjustment calculus, such as the transformation of multiple integrals.

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## 1. INTRODUCTION

This study represents a continuation of efforts aimed at developing the least-squares theory and results in a purely geometrical manner. It is based on [Blaha, 1984], abbreviated here as [B]. One of the important limitations listed in this reference is the full column rank of the design matrix of the parametric adjustment. As its title suggests, the present analysis is intended to extend the scope of [B] by bridging this limitation. Undoubtedly, most or all of the others can similarly be bridged, one by one.

In both [B] and the present study, the various finite-dimensional spaces are Riemannian, where the metric (or line element) is expressible by means of a symmetric covariant tensor of second order. These spaces are considered only within an infinitesimal neighborhood of the point called P, contained by all of them and corresponding to the point of Taylor expansion in the parent least-squares (L.S.) problem. In accordance with most adjustment applications, where all except the first (linear) terms in the expansion are ignored, the expanded L.S. problem as treated here is linear, whether its original version was linear or not. The same statement applies, of course, also to [B], which expressly avoided nonlinear adjustment. The above spaces can thus be regarded as Euclidian (or flat), which, by definition, can be described over a finite region in Cartesian coordinates.

In view of the foregoing, the term "surface" used in [B] means actually "hyperplane", since it is not limited to two dimensions and since it is intrinsically a flat space. However, references to surface will be retained for convenience, allowing for an easy transfer of familiar terminology, e.g. from [Hotine, 1969], concerned with two-dimensional surfaces embedded in a three-dimensional space. This terminology should present no confusion since we know that such surfaces will not be two-dimensional in general; as a special case, the model "surface" could even be one-dimensional, reducing here to a straight line through P, of which only a small segment in the neighborhood of P would be of interest.

From the philosophical standpoint, the geometrical analysis of the rank deficient parametric adjustment typifies the notion that could be dubbed an "orthonormalization of the least-squares universe". In this sense, the fundamental building blocks consist of orthonormal vectors emanating from the

point P. These vectors are considered as fixed entities, and their subsets span the spaces and surfaces linked to the L.S. setup. Their components are separated into contravariant (denoted by an upper index) and covariant (denoted by a lower index) in accordance with the principles of tensor analysis, and are regarded as point functions at P.

From the L.S. setup to the desired results including the variance-covariance and the weight matrices, the standard adjustment quantities can be represented by first- and second-order tensors. It is thus possible to express them in terms of the components of the above orthonormal vectors, which is indeed the main feature of both the previous and the current studies. This procedure, called in [B] "vectorization" of tensors, allows for an easy and clearcut geometrical interpretation of the L.S. process. As a by-product, it circumvents the need to verify, at various stages of the development, whether certain "objects" are tensors or not, which under different circumstances would be done by checking whether the tensor transformation law applies.

The vectorization of the metric tensor (in mathematical literature also called the covariant metric tensor or the fundamental metric tensor) and the associated metric tensor (also called the contravariant metric tensor or the conjugate metric tensor) is especially relevant in view of a complete treatment of adjustment problems. By virtue of the vectorization, the "propagation" of these tensors was shown in [B] to fit perfectly the variance-covariance propagation law and even to establish a weight propagation law. This indicates that the vectorization is a tool whose potential is readily exploited in the tensorial environment. Without the use of tensor structure, with its contravariant and covariant versions of the metric tensor, the side-by-side derivation of these propagation laws would have been more difficult if not impossible. The opportunity to obtain the propagated variance-covariances and weights as a coherent part of the geometrical development constitutes a strong motivation for using tensor structure in the description, treatment, and analysis of various L.S. methods and their properties.

## 2. GEOMETRICAL SETUP

### 2.1 General Background

In pursuing the avenue of relating, for example, associated metric tensors to variance-covariance matrices, one soon encounters rank-deficient symmetric contravariant tensors which behave as the "regular" associated metric tensors at  $P$ , but only when applied to tensors restricted to a given surface embedded in some original space. In other words, they can be regarded as associated metric tensors in spaces of lower dimensions than the original space, but expressed in the full-dimensional form of the original space. They could be called "restrictive associated metric tensors", "defective associated metric tensors", etc. However, for the reason explained below they will be called "necessary associated metric tensors". A similar description applies also when relating metric tensors to weight matrices. In this case, the pertinent rank-deficient covariant tensors will be called "necessary metric tensors".

Suppose that an (original)  $n$ -dimensional space is spanned by  $n$  orthonormal vectors denoted as  $\ell, j, \dots, \nu, \dots$  belonging to the point  $P$ , and that a  $u'$ -dimensional surface of interest embedded in this space is spanned by  $u'$  of these vectors, namely  $\ell, j, \dots$ . The associated metric tensor is then given as

$$g^{rs} = \ell^r \ell^s + j^r j^s + \dots + \nu^r \nu^s + \dots,$$

while the necessary associated metric tensor is

$$g'^{rs} = \ell^r \ell^s + j^r j^s + \dots,$$

where the indices  $r$  and  $s$ , identifying the space components, range between 1 and  $n$ . Both of these tensors raise the index of  $dx'_s$  representing the covariant components of an arbitrary vector  $dx'$  lying in the surface. However, the use of  $g^{rs}$  is sufficient, but not necessary. The tensor of the lowest rank that can accomplish this is  $g'^{rs}$ , both necessary and sufficient, hence the attribute "necessary".

In the present context, the  $n$ -dimensional space spanned by  $\ell, j, \dots, \nu, \dots$  is called "observational space", the  $u'$ -dimensional surface spanned by  $\ell, j, \dots$  is called "model surface", and a further  $n''$ -dimensional surface spanned by  $\nu, \dots$  is called "error surface". The error surface is an orthocomplement of

the model surface in the observational space (all considered at P). The necessary associated metric tensor for the error surface is written as

$$g^{rs} = \nu^r \nu^s + \dots,$$

so that

$$g^{rs} = g'^{rs} + g''^{rs}. \quad (1)$$

This relationship indicates a close analogy to

$$dx^r = dx'^r + dx''^r, \quad (2)$$

where the vector  $dx$  in the observational space is decomposed into two orthogonal vectors,  $dx'$  lying in the model surface and  $dx''$  lying in the error surface. The lengths (tensor invariants) of these three vectors are respectively  $ds$ ,  $ds'$ , and  $ds''$ . All of the above tensor equations could, of course, be written with lower indices instead, in which case the term "associated" would be dropped.

Similar to [B], the set of contravariant components  $dx^r$  is considered to represent observations in an adjustment model after linearization. It is decomposed into  $dx_{(1)}^r$  and  $dx_{(2)}^r$ , where the first set belongs to a vector  $dx_{(1)}$  lying in the model surface and the second set completes the system of equations  $dx^r = dx_{(1)}^r + dx_{(2)}^r$ . (The restriction of  $dx_{(1)}$  to the model surface can be thought of as the geometrical equivalent of a consistent model relationship between the observables and the parameters.) In general, there exists an infinite number of solutions for  $dx_{(1)}^r$ . However, if the quadratic form  $dx_{(2)}^s g_{sr} dx_{(2)}^r$ , i.e., the square of the length of the vector  $dx_{(2)}$ , should be a minimum,  $dx_{(2)}$  must be orthogonal to the model surface and lies, therefore, in the error surface. Consequently, the vectors  $dx_{(1)}$  and  $dx_{(2)}$  become unique, such that  $dx_{(1)} \equiv dx'$  and  $dx_{(2)} \equiv dx''$ . Furthermore, the quadratic form

$$ds''^2 = dx''^s g_{sr} dx''^r = \text{minimum} \quad (3)$$

depicts the standard L.S. criterion written in adjustment notations as  $V^T P V = \hat{\epsilon}^T P \hat{\epsilon} = \text{minimum}$ , where  $V \equiv -\hat{\epsilon}$  represents the residuals (corresponding here to  $-dx''^r$ ) and  $P$ , not to be confused with the point  $P$ , is the weight matrix of observations (corresponding to  $g_{sr}$ ). The latter is defined as  $P = C^{-1}$ , where  $C$  is a given variance-covariance matrix of observations (corresponding to  $g^{rs}$ ).

In expanding the present terminology,  $dx$  is called "observational vector",  $dx'$  is called "model vector", and  $dx''$  is called "error vector",  $dx'$  being an orthogonal projection of  $dx$  on the model surface and  $dx''$  being an orthogonal projection of  $dx$  on the error surface. The L.S. criterion is thus seen in the geometrical context as minimizing the length of the error vector in the observational space metricized by the weight matrix of observations. The foregoing discussion is general, applicable to the full-rank and the rank-deficient adjustments alike. The next section highlights the geometrical distinction between the two kinds of adjustments.

## 2.2 Universal Space and its Partition

The vectors  $dx$ ,  $dx'$ , and  $dx''$  as introduced above are illustrated schematically in Fig. 1. The model vector  $dx'$  is also denoted as  $du'$  depending on the coordinate system used to express its components. The geometry of these vectors resembles that of Fig. 1 in [B], except that the observational space in the present figure is two-dimensional,  $n=2$  (instead of three-dimensional), and the model surface is one-dimensional,  $u'=1$  (instead of two-dimensional). The error surface, in [B] called "second surface", is one-dimensional in both illustrations,  $n''=1$ .

In considering rank-deficient adjustment problems, we define the model surface as embedded not only in the observational space, but also in a new,  $u$ -dimensional "parametric space". Compared to its full-rank counterpart, the dimensionality of this geometrical setup is increased by  $u''$ , where  $u''$  designates the dimensions of a new subspace called "extension surface". The latter is an orthocomplement of the model surface in the parametric space, and is defined to be orthogonal to the observational space. Accordingly, the complete geometrical configuration must be presented in an all-encompassing  $N$ -dimensional "universal space", where  $N=n+u''=n''+u$ , with  $u=u'+u''$ . In the illustration of Fig. 1, the dimensions not listed above are  $u''=1$ ,  $u=2$ , and  $N=3$ .

If the extension surface were absent, i.e., if  $u''=0$ , the situation of Fig. 1 would correspond to a full-rank L.S. setup. The universal space would be identical to the observational space and the parametric space would be identical to the model surface, hence  $N=n$  and  $u=u'$ . The model vector  $du'$  lying in the  $u'$ -dimensional model surface would then be expressed in model-surface coordinates,

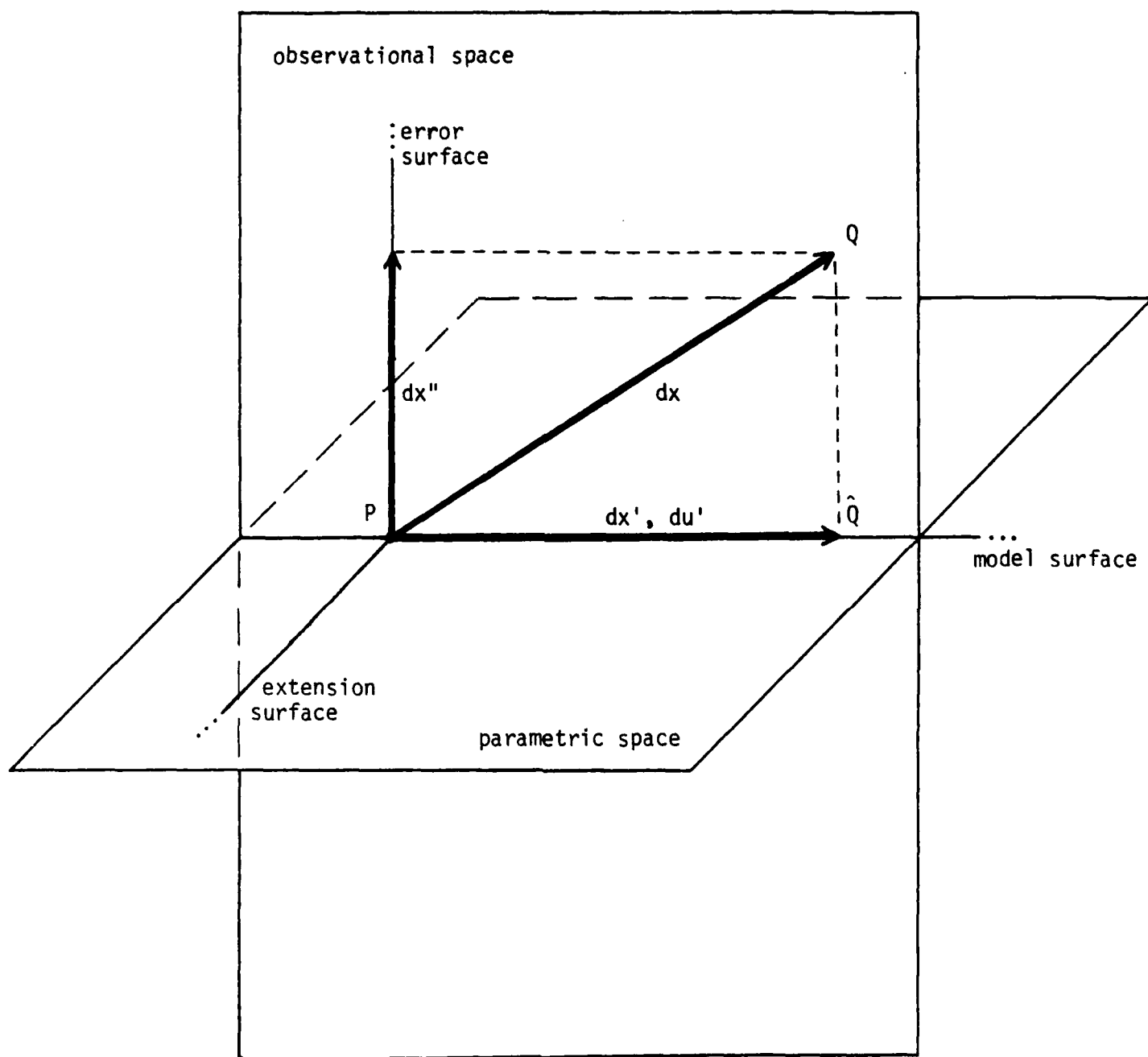


Fig. 1

Symbolic representation, in the N-dimensional universal space (here  $N = 3$ ), of the observational vector  $dx$  and other related vectors in the geometrical setup corresponding to the rank-deficient parametric adjustment

i.e., it would have  $u'$  components of either kind (contravariant and covariant). On the other hand, the rank-deficient L.S. setup corresponds to  $u'' > 0$  and thus  $u > u'$ , the model surface being a subspace of the parametric space as stated in the preceding paragraph. In the geometrical representation, the rank-deficient L.S. setup is distinguished from its full-rank counterpart by the fact that the model vector  $du'$  lying in the  $u'$ -dimensional model surface is expressed in parametric-space coordinates rather than in model-surface coordinates, i.e., it has  $u > u'$  components of either kind.

The universal-space configuration corresponding to the rank-deficient L.S. setup is elucidated via orthonormal vectors emanating from P. In particular, the universal space is spanned by  $N$  orthonormal vectors  $\nu, \dots, \ell, j, \dots, t, \dots$ , giving rise to its partition as follows:

$$\begin{array}{ccc}
 n'' + u' = n \text{ -dimensional} & & u' + u'' = u \text{ -dimensional} \\
 \text{observational space} & & \text{parametric space} \\
 \nu, \dots, \ell, j, \dots, t, \dots & & (4) \\
 n'' \text{ -dimensional} & & u' \text{ -dimensional} & & u'' \text{ -dimensional} \\
 \text{error surface} & & \text{model surface} & & \text{extension surface}
 \end{array}$$

The error surface is confirmed to be an orthocomplement of the model surface in the observational space, while the extension surface, spanned by the  $u''$  orthonormal vectors  $t, \dots$ , is confirmed to be an orthocomplement of the model surface in the parametric space. The extension surface could further be viewed as an orthocomplement of the observational space in the universal space, etc. Moreover, the model surface is seen to be an intersection of the observational and the parametric spaces.

The partition of the universal space in (4) indicates that more than one coordinate system may become involved at various stages of the development. In addition to a coordinate system associated with the universal space itself, each of the following spaces and surfaces are endowed with a coordinate system symbolized by braces:



observational space .....  $\{x^r\}$ ,  $r=1,2,\dots,n$ ; (5a)

parametric space .....  $\{u^\alpha\}$ ,  $\alpha=1,2,\dots,u$ ; (5b)

model surface .....  $\{u^L\}$ ,  $L=1,2,\dots,u'$ ; (5c)

extension surface .....  $\{w^\Lambda\}$ ,  $\Lambda=1,2,\dots,u''$ . (5d)

The systems (5a,b,d) will serve in the next chapter to resolve the rank-deficient L.S. setup. The systems (5c,d) will be used in the next section as a stepping stone toward expressing a "rank-deficient design tensor" in terms of partial derivatives. According to an earlier statement, the full-rank L.S. setup would be characterized by an identity between the systems (5b) and (5c), and by the absence of the system (5d).

We now present, in Table 1, tensor quantities expressed in the coordinate systems (5a,b,d). They include first- and second-order contravariant tensors, and second-order mixed tensors, all representing point functions at P. One can imagine first- and second-order covariant tensors added to the table following the pattern of its first two parts with all the indices lowered. Great many tensor relations can be derived with the aid of this table, such as equations (1) and (2) which can be read directly, or more complex expressions which can be formed through tensor contractions. Thus, Table 1 will be relevant in much of the geometrical development in this study.

The arrangement of spaces and surfaces in Table 1 conforms to their representation in (4). Due to the vectorized formulation of all the tensors, the latter are automatically classified in two respects, namely, according to the space or surface in which they exist as geometrical objects, and according to the coordinate system used to express their individual components. In considering the first classification, one can write the identity  $dx' \equiv du'$ , for example, stating that the vectors  $dx'$  and  $du'$  are one and the same geometrical object, represented by "a" units along the vector  $\ell$ , "b" units along the vector  $j$ , etc. With regard to the second classification, the component sets  $dx'^r$  and  $du'^\alpha$  are given a fundamentally different adjustment interpretation from one another. With both sets referring to a linearized model, the former represents the adjusted observations and the latter represents the adjusted parameters.

	observational space	parametric space
	error surface	model surface      extension surface
$dx^r =$	$q \nu^r + \dots + a l^r + b j^r + \dots$	
$dx'^r =$		$a l^r + b j^r + \dots$
$dx''^r =$	$q \nu^r + \dots$	
$du^\alpha =$		$a l^\alpha + b j^\alpha + \dots + z t^\alpha + \dots$
$du'^\alpha =$		$a l^\alpha + b j^\alpha + \dots$
$du''^\alpha =$		$z t^\alpha + \dots$
$dw^\Lambda =$		$z t^\Lambda + \dots$
<hr/>		
$g^{rs} =$	$\nu^r \nu^s + \dots + l^r l^s + j^r j^s + \dots$	
$g'^{rs} =$		$l^r l^s + j^r j^s + \dots$
$g''^{rs} =$	$\nu^r \nu^s + \dots$	
$a^{\alpha\beta} =$		$l^\alpha l^\beta + j^\alpha j^\beta + \dots + t^\alpha t^\beta + \dots$
$a'^{\alpha\beta} =$		$l^\alpha l^\beta + j^\alpha j^\beta + \dots$
$a''^{\alpha\beta} =$		$t^\alpha t^\beta + \dots$
$k^{\Lambda\Omega} =$		$t^\Lambda t^\Omega + \dots$
<hr/>		
$A_\alpha^r =$	$l^r l_\alpha + j^r j_\alpha + \dots$	
$Q_r^\alpha =$	$l^\alpha l_r + j^\alpha j_r + \dots$	
$\bar{\Lambda}_\Lambda^\alpha =$		$t^\alpha t_\Lambda + \dots$
$\bar{Q}_\alpha^\Lambda =$		$t^\Lambda t_\alpha + \dots$

Table 1  
First- and second-order tensors in the partitioned universal space

### 2.3 Rank-Deficient Design Tensor

In this section, the tensor  $A_{\alpha}^r$  of Table 1, called the rank-deficient design tensor, will be developed in terms of partial derivatives. This tensor will thus be shown as structurally similar to its counterpart in [B], which reflected the actual formation of the design matrix in the full-rank context. Indeed, the design matrix is constructed, in adjustment calculus, through first-order partial derivatives of the observables with respect to the parameters whether it is full-rank or rank-deficient. Showing that the design tensor follows the same pattern further supports the analogy between the adjustment theory and geometry. Moreover, this knowledge paves the way for the treatment of nonlinear (and non-linearized) adjustment models containing higher-order derivatives of the observables with respect to the parameters.

The initial step in the current development makes use of the coordinate systems (5c,d). Merged together, these systems serve to define an interim (overbarred) coordinate system for the parametric space,  $\{\bar{u}^{\alpha}\} = \{u^L, w^{\Lambda}\}$ . An important property of such an arrangement is that the interim system is normal, in the sense that the last  $u''$  components, contravariant and covariant, of vectors lying in the model surface are zero, as are the first  $u'$  components of vectors lying in the extension surface. It is thus a higher-dimensional analogue of the normal coordinate systems in three dimensions as described in Chapter 15 of [Hotine, 1969].

The above generalized statement will now be proved in detail. In analogy to the N-surfaces of [Hotine, 1969], where any such (two dimensional) surface is characterized by a constant value of the third spatial coordinate (N), a  $u'$ -dimensional surface embedded in a  $u$  dimensional space, where  $u = u' + u''$ , is now characterized by  $u''$  constant coordinates. In particular, the  $u$ -dimensional space is the parametric space, the  $u''$  coordinates held constant are  $w^{\Lambda}$ , and the  $u'$  dimensional surface is the model surface with the (variable) coordinates  $u^L$ . Since the latter are also the first  $u'$  of the space coordinates  $\bar{u}^{\alpha}$ , any displacement vector ( $du'$ ) on the surface expressed by the surface contravariant components  $du'^L$  has  $du'^L$  also as the first  $u'$  space contravariant components, the last  $u''$  such components being zero. This relationship is symbolized by  $du'^{\alpha} = (du'^L, 0)$ . Applied to the orthonormal vectors  $\ell, j, \dots$ , it yields

$$\ell^{\alpha} = (\ell^L, 0), \quad j^{\alpha} = (j^L, 0), \dots \quad (6a)$$

Since  $\bar{t}_\alpha \bar{t}^\alpha = 0$ ,  $\bar{t}_\alpha \bar{j}^\alpha = 0$ , ... represent a nonsingular system of  $u'$  equations, the  $u'$  unknowns  $\bar{t}_L$ ,  $L=1,2,\dots,u'$  are unique, all equal to zero. This is true for all  $u''$  orthonormal vectors  $t$ , ... spanning the extension surface. However, little could be said about the covariant components  $\bar{t}_\alpha$ ,  $\bar{j}_\alpha$ , ... if the last  $u''$  space coordinates were not at the same time the extension-surface coordinates. But since  $w^\Lambda$  have this property by construction, in analogy to the preceding paragraph we first deduce that

$$\bar{t}^\alpha = (0, t^\Lambda) , \dots , \quad (6b)$$

where the extension-surface contravariant components of the vectors  $t$ , ... are also their last  $u''$  space contravariant components. Now  $\bar{t}_\alpha \bar{t}^\alpha = 0$ , ... represent a nonsingular system of  $u''$  equations, whereby the  $u''$  unknowns  $\bar{t}_\Lambda$ ,  $\Lambda=1,2,\dots,u''$  are unique, all equal to zero. This is true for all  $u'$  orthonormal vectors  $t$ ,  $j$ , ... spanning the model surface.

We next contract the partially known  $\bar{t}_\alpha$  with  $\bar{t}^\alpha$ ,  $\bar{j}^\alpha$ , ... from (6a) and obtain a nonsingular system of  $u'$  equations in  $u'$  unknowns  $\bar{t}_L$ ,  $L=1,2,\dots,u'$ . However, in considering the model surface, the same equations hold true also without the overbar. Since this argument again applies to all of  $t$ ,  $j$ , ... , it follows that

$$\bar{t}_\alpha = (t_L, 0) , \quad \bar{j}_\alpha = (j_L, 0) , \dots , \quad (7a)$$

where  $t_L$ ,  $j_L$ , ... represent the model-surface components. Finally, upon contracting the partially known  $\bar{t}_\alpha$  with  $\bar{t}^\alpha$ , ... from (6b), we obtain a nonsingular system of  $u''$  equations in  $u''$  unknowns, eventually resulting in

$$\bar{t}_\alpha = (0, t_\Lambda) , \dots , \quad (7b)$$

where  $t_\Lambda$ , ... represent the extension-surface components. The outcome (6a)-(7b) is possible only because the two groups of coordinates forming the interim system are surface coordinates in their own right, and because the two surfaces are orthocomplements of each other in the underlying parametric space.

To complete the discussion concerned with the interim coordinate system, we note that the associated metric tensor for the parametric space is

$$g^{\alpha\beta} = \bar{t}^\alpha \bar{t}^\beta + \bar{j}^\alpha \bar{j}^\beta + \dots + \bar{t}^\alpha \bar{t}^\beta + \dots ,$$

or

$$g^{\alpha\beta} = (a^{LM}, k^{\Lambda\Omega}) \quad (8a)$$

where  $a^{LM}$  and  $k^{\Lambda\Omega}$  are the associated metric tensors for the model surface and for the extension surface, respectively. This follows from the standard relations

$$a^{LM} = \ell^L \ell^M + j^L j^M + \dots,$$

$$k^{\Lambda\Omega} = t^{\Lambda} t^{\Omega} + \dots,$$

generalized from two to any dimensions. The symbolism in (8a) indicates that in matrix notations,  $a^{LM}$  and  $k^{\Lambda\Omega}$  would form the diagonal submatrices, while the off-diagonal submatrices would be zero as one of the basic characteristics of a normal system. In the same way, but with all the indices lowered, the metric tensor for the parametric space can be written as

$$\bar{g}_{\beta\alpha} = (a_{ML}, k_{\Omega\Lambda}) \quad (8b)$$

In considering that the model surface is embedded in the observational space, we can express each of the  $x^r$  observational-space coordinates describing this surface as some function of the model-surface coordinates  $u^L$ , namely

$$x^r = x^r(u^L); \quad r=1,2,\dots,n; \quad L=1,2,\dots,u'.$$

The ordinary rule of differentiation for this system of equations yields the following linear relations between the observational space components  $dx'^r$  and the model-surface components  $du'^L$  of the vector  $dx' \equiv du'$  lying in the model surface:

$$dx'^r = (\partial x^r / \partial u^L) du'^L \quad (9a)$$

If this formula, relating the space and the surface components of such vectors, is in turn applied to  $\ell, j, \dots$ , it follows that

$$\partial x^r / \partial u^L = A_L^r = \ell^r \ell_L + j^r j_L + \dots \quad (9b)$$

where use has been made of  $\ell^L \ell_M + j^L j_M + \dots = \delta_M^L$ . Equations (9a,b) appear in a similar form in [8], and represent a higher dimensional analogue of the formulas found in [Hotine, 1969], applicable to a two-dimensional surface embedded in a three-dimensional space.

The vectorized formulation of  $\partial x^r / \partial u^L$  seen above is based on the fact that the model surface (with coordinates  $u^L$ ) is embedded in the observational space (with coordinates  $x^r$ ). On the other hand, the extension surface (with coordinates  $w^A$ ) has no dimension in common with the observational space, hence  $\partial x^r / \partial w^A = 0$ . When joined together, these two sets of partial derivatives yield

$$(\partial x^r / \partial u^L, \partial x^r / \partial w^A) = i^r(i_L, 0) + j^r(j_L, 0) + \dots$$

But in view of the interim system, the left-hand side above can be written in a compact form as  $\partial x^r / \partial u^\alpha$ , while the components within the parentheses on the right hand side are  $i_\alpha, j_\alpha, \dots$  by (7a). This relation can thus be transcribed in tensor notations as

$$\partial x^r / \partial u^\alpha = i^r i_\alpha + j^r j_\alpha + \dots \quad (10)$$

That (10) is a tensor equation valid in conjunction with any parametric-space coordinates can be confirmed through the transformation formulas

$$i_\alpha = (\partial u^\beta / \partial u^\alpha) i_\beta, \quad j_\alpha = (\partial u^\beta / \partial u^\alpha) j_\beta, \dots$$

where the coordinates  $u^\beta$  belong to the general system (5b). These relations are substituted in (10), the latter is contracted by  $\partial u^\alpha / \partial u^\gamma$ , use is made of  $(\partial u^\beta / \partial u^\alpha)(\partial u^\alpha / \partial u^\gamma) = \partial u^\beta / \partial u^\gamma = \delta_\gamma^\beta$  and, finally, the index  $\gamma$  is substituted for by  $\alpha$ . The result, where the interim system is no longer needed, reads

$$\partial x^r / \partial u^\alpha = A_\alpha^r i_\alpha + j_\alpha + \dots \quad (11)$$

Equation (11) represents the complete description of the rank deficient design tensor, whose full rank counterpart played a central role in [8]. Although this reference displayed the full rank design tensor in the form (11), in the present notations it would be properly represented by (9b). In considering that the model surface is also embedded in the parametric space, we could introduce a relation similar to (9b), where the symbol  $x$  would be replaced by  $u$  and all the indices  $r$  would be replaced by  $\alpha$ . Equations (9b) and (11) together with this new relation would then confirm, through tensor contractions, the validity of the chain rule  $\partial x^r / \partial u^L = (\partial x^r / \partial u^\alpha)(\partial u^\alpha / \partial u^L)$ .

As their structure reveals, the partial derivatives forming the design tensor in (11) transform the contravariant components of vectors lying in the model surface from the parametric space coordinate system  $\{u^\alpha\}$  to the observational space coordinate system  $\{x^r\}$ . This property, fundamental to the

treatment of a rank-deficient adjustment model via an isomorphic geometrical setup, is utilized in the form

$$dx'^R = A_{\alpha}^R du'^{\alpha}, \quad (12)$$

where the notations  $du'$  and  $dx'$  designate the same vector as pointed out earlier. This equation was already implied by Table 1, where the design tensor was defined in anticipation of the result (11). As a matter of interest, we note that a relation similar to (12) could be written with  $du$  replacing  $du'$ , where  $du$  is a vector from the parametric space such that  $du^{\alpha} = du'^{\alpha} + du''^{\alpha}$  as suggested by Table 1. This stems from the fact that the components  $du''^{\alpha}$  of a vector lying in the extension surface yield zero when contracted with  $A_{\alpha}^R$ .

In matrix notations, the rank deficient design tensor  $A_{\alpha}^R$  is written as  $A$ , the familiar design matrix. Following the conventions of [8] for second-order tensors, the first and the second indices refer to rows and columns, respectively; in the case of a mixed tensor such as  $A_{\alpha}^R$ , the contravariant index is considered as its first and the covariant index as its second. By virtue of (11), the design matrix can be decomposed into a product of two matrices as follows:

$$[\partial x^R / \partial u^{\alpha}] = A = \begin{matrix} & & & L^{*T} \\ (n \times u) & (n \times u') & (u' \times u) \end{matrix} F, \quad (13a)$$

where

$$F = [[\ell^R][j^R] \dots], \quad L^* = [[\ell_{\alpha}][j_{\alpha}] \dots], \quad (13b,c)$$

and where  $[\ell^R]$ , etc., represent column vectors. The matrix  $F$  is rectangular, of dimensions  $(n \times u')$  and the full column rank  $u'$ , and the matrix  $L^{*T}$  is rectangular of dimensions  $(u' \times u)$  and the full row rank  $u'$ . In the case of (9b), the number of columns in the latter matrix would reduce to  $u'$ , making it regular, i.e., square and nonsingular, and resulting in the full rank design matrix particular to [8]. Although both  $F$  and  $L^{*T}$  considered at the present have the full (column or row) rank individually, their product in (13a) is rank deficient, the rank deficiency being  $u - u' - u''$ . The design matrix  $A$  of dimensions  $(n \times u)$  and rank  $u'$ , where  $n < u$ , characterizes the rank deficient parametric adjustment. We note that if  $n = u$ , the model surface would coincide with the observational space and no L.S. adjustment would take place.

Similar to the full-rank case, the advantages of the geometrical approach to the analysis of the rank-deficient design matrix are readily apparent:

- 1) The design matrix is decomposed into the product of two constituent matrices;
- 2) The latter are written in terms of orthonormal vectors, the elementary geometrical entities; and
- 3) These vectors are the same for both constituent matrices, only the type of their components differs.

The decomposition of the rank-deficient design matrix  $A$  offers geometrical insight that cannot be gathered from algebraic considerations usually based on the column space of  $A$  in its original form, not on spaces associated with the more elementary matrices seen above.

#### 2.4 Propagation of Contravariant and Covariant Metric Tensors

We have seen that the  $u'$ -dimensional model surface, spanned by  $u'$  orthonormal vectors  $\ell, j, \dots$ , is embedded in the  $n$ -dimensional observational space, spanned by  $n$  orthonormal vectors  $\ell, j, \dots, \nu, \dots$ . The model surface is also embedded in the  $u$ -dimensional parametric space, spanned by  $u$  orthonormal vectors  $\ell, j, \dots, t, \dots$ . The rank deficiency is rooted in the fact that the vectors lying in the model surface, including  $\ell, j, \dots$  themselves, are expressed by  $u$  parametric-space components ( $u > u'$ ) rather than by  $u'$  model-surface components as in the full-rank setup of [8].

The above reference established a perfect correspondence between the propagation of associated metric tensors and the variance-covariance propagation law, and between the propagation of metric tensors and the weight propagation law. For the most part, however, this isomorphism was proven in conjunction with vectors lying in the model surface and expressed through the model-surface components. The scope of the proof will now be extended by showing that the same correspondence can be established if the parametric-space components are used instead. Similar to [8], below we list relationships between vector components, and next to them (separated by dots) formulate the corresponding



relationships between contravariant or covariant metric tensors. Any of these relationships can be written at once with the aid of Table 1.

In particular, this table (including its extensions to the covariant components) yields

$$du'^{\alpha} = Q_r^{\alpha} dx'^r = Q_r^{\alpha} dx^r \dots \dots a'^{\alpha\beta} = Q_r^{\alpha} g'^{rs} Q_s^{\beta} = Q_r^{\alpha} g^{rs} Q_s^{\beta} . \quad (14a,b)$$

$$du'_{\beta} = A_{\beta}^s dx'_s = A_{\beta}^s dx_s \dots \dots a'_{\beta\alpha} = A_{\beta}^s g'_{sr} A_{\alpha}^r = A_{\beta}^s g_{sr} A_{\alpha}^r . \quad (14c,d)$$

Equations (14a-d) depict what could be termed the situation n-u', indicating that components in n dimensions are transformed into components in u' dimensions. Similarly, we deduce

$$dx'^r = A_{\alpha}^r du'^{\alpha} = A_{\alpha}^r du^{\alpha} \dots \dots g'^{rs} = A_{\alpha}^r a'^{\alpha\beta} A_{\beta}^s = A_{\alpha}^r a^{\alpha\beta} A_{\beta}^s . \quad (15a,b)$$

$$dx'_s = Q_s^{\beta} du'_{\beta} = Q_s^{\beta} du_{\beta} \dots \dots g'_{sr} = Q_s^{\beta} a'_{\beta\alpha} Q_r^{\alpha} = Q_s^{\beta} a_{\beta\alpha} Q_r^{\alpha} . \quad (15c,d)$$

Equations (15a-d) depict the situation called u'-n. The (associated) metric tensors attributed a prime are the "necessary" tensors as described earlier. Table 1 shows that the Q-tensor is related to the A-tensor through

$$Q_r^{\alpha} = a'^{\alpha\beta} A_{\beta}^s g_{sr} . \quad (15e)$$

where the a- and/or g-tensors could be replaced by their primed counterparts.

We present an additional case of interest, called the situation u'-m, which is conceptually quite similar to the situation u'-n reflected in (15a-d). It is based on an m-dimensional "functional space", which is yet another space containing the model surface, hence m>u'. This space could be thought of as embedded in the observational space, identical to it, or containing it, corresponding to m<n, m=n, or m>n, respectively. Much like the observational space, the functional space intersects the parametric space in the model surface, and has no dimension in common with the extension surface. We could thus imagine it as replacing temporarily the observational space, in which case all the tensors in Table 1 having r and/or s as indices would be attributed the symbol ~, and the lower case Roman letters themselves would be replaced by another kind of indices whose range would extend from 1 to m (instead of 1 to n). Except for these changes, the basic relations (15a-d) could be rewritten as they stand, and the same applies for the connecting equation (15e). One could

even retrace the steps (9a)-(11) and represent the new  $\tilde{A}$ -tensor in terms of partial derivatives by rewriting (11) with the same notational changes. Clearly, the rank of the  $\tilde{A}$ - and  $\tilde{Q}$ -tensors would be  $u'$  by construction, the same as the rank of their counterparts in (15a-e).

If only the first  $m_1$  of the  $m$  functional-space vector components were of any relevance, the computation of the corresponding (associated) metric tensor would likewise be limited to its first  $m_1 \times m_1$  components. If, similarly, only  $m_1 \leq u'$  functional-space components of the  $\tilde{A}$ - and/or  $\tilde{Q}$ -tensors were known, one could imagine their range extended through  $m > u'$  to ensure the validity of the relationships described in the previous paragraph. The actual components would subsequently be computed only to within the first  $m_1$  for the desired vector and to within the first  $m_1 \times m_1$  for the corresponding (associated) metric tensor, i.e., the parts imagined for the sake of the theory would be disregarded. We can thus conclude that whether the number of functional-space components in the  $\tilde{A}$ - and/or  $\tilde{Q}$ -tensors is larger than  $u'$  or not, the resulting vector components and the corresponding (associated) metric tensor components follow the pattern of (15a-d).

In order to highlight the isomorphism between the propagation of contravariant or covariant metric tensors in the geometrical context and the propagation of variance-covariance or weight matrices in the adjustment context, we transcribe (14a)-(15d) in matrix notations. In so doing, we adopt the "traditional identification" of [B], whereby associated metric tensors correspond to variance-covariance matrices and metric tensors correspond to weight matrices. Due to its adjustment appeal, this identification was preferred in [B] to the "new identification" with reversed correspondences, although both identifications were shown to lead to identical results. In concentrating on the variance-covariance matrices first, we transcribe (14a,b) and (15a,b) as

$$du' = Qdx' = Qdx \dots a' = Qg'Q^T = QgQ^T, \quad (14'a,b)$$

$$dx' = Adu' = Adu \dots g' = Aa'A^T = AaA^T. \quad (15'a,b)$$

The connection between  $Q$  and  $A$  is provided by

$$Q = aA^T g^*, \quad (15'e)$$

which follows from (15e), and where the matrices  $a$  and/or  $g^*$  could be attributed a prime.

As can be gathered from these relations, all of the first- and second-order contravariant or mixed tensors keep their symbols also in matrix notations, except that the indices are dropped. This simplifies the notational conventions used in [B], where a new set of symbols was introduced in matrix notations in order to conform more closely to adjustment notations. No confusion should arise from a dual role of the symbol  $dx$ , for example, which in one context designates a geometrical object associated with the component set  $dx^F$ , and in the other expresses the set  $dx^F$  as a column vector. In (14'a,b) and (15'a,b), the variance-covariance matrices corresponding to the (column) vectors  $dx$ ,  $dx'$ ,  $du$ , and  $du'$  are  $g$ ,  $g'$ ,  $a$ , and  $a'$ , respectively. These equations express the familiar variance-covariance propagation law of adjustment calculus, valid whether or not any of the matrices are rank-deficient. The case described above in conjunction with the functional space would be included in (15'a,b), except that  $dx'$ ,  $A$ , and  $g'$  would be attributed the symbol  $\tilde{\phantom{x}}$ . It would correspond to the variance-covariance propagation applied to linear functions of  $du'$ . In retrospect, this fact provided the motivation for the term "functional space".

According to the discussion that followed (15e), the matrix  $\tilde{A}$  can be written as

$$\tilde{A} = \tilde{F}L^{*T},$$

which is the functional-space version of (13a). As a matter of interest, we also present this matrix in the form

$$\tilde{A} = RA, \quad R = \tilde{F}F^Tg^*,$$

where  $R$  has the dimensions  $(m \times n)$  and the rank  $u'$ . The second of the above formulas has been obtained from both versions of (13a) together with the identity  $P^Tg^*F-I$ , but in the present context it is not needed. Although the situation  $u' = m$  is illustrated sufficiently well by equations (15'a,b) in their functional-space version, we can substitute  $\tilde{A} = RA$  in the latter and write

$$d\tilde{x}' = Rdx' \quad \dots \quad \tilde{g}' = Rg'R^T,$$

where use has been made of equations (15'a,b) in their original version. This formulation further highlights the isomorphism between the associated metric tensors and the variance-covariance matrices.

The first- and second-order covariant tensors keep their symbols in matrix notations as well (with the indices dropped), but are attributed \* to be distinguished from their contravariant counterparts. The relations (14c,d) and (15c,d) are thus transcribed as

$$du^{*'} = A^T dx^{*'} = A^T dx^* \dots\dots a^{*'} = A^T g^{*'} A = A^T g^* A , \quad (14'c,d)$$

$$dx^{*'} = Q^T du^{*'} = Q^T du^* \dots\dots g^{*'} = Q^T a^{*'} Q = Q^T a^* Q . \quad (15'c,d)$$

The weight matrices corresponding to the adjustment vectors  $dx$ ,  $dx'$ ,  $du$ , and  $du'$  are  $g^*$ ,  $g^{*'}$ ,  $a^*$ , and  $a^{*'}$ , respectively. In (14'c,d) and (15'c,d), the weight matrices are related to the covariant version of the pertinent vectors through a structure resembling (14'a,b) and (15'a,b) and characterizing the weight propagation law.

Equations (15'c,d) with  $\sim$  attributed to  $dx^{*'}$ ,  $Q$ , and  $g^{*'}$  reflect the functional-space context. Equation (15'e) is valid here in the form

$$\tilde{Q} = a \tilde{A}^T \tilde{g}^* ,$$

where the matrices  $a$  and/or  $\tilde{g}^*$  could again be attributed a prime. In paralleling the development of the previous paragraph, we could show that

$$\tilde{Q} = Q S , \quad S = F \tilde{F}^T \tilde{g}^* ,$$

where  $S$  has the dimensions  $(n \times m)$  and the rank  $u'$ . Finally, in utilizing both versions of (15'c,d) in conjunction with the first formula above, we obtain

$$d\tilde{x}^{*' } = S^T dx^{*' } \dots\dots \tilde{g}^{*' } = S^T g^{*' } S ,$$

which further confirms the pattern characterizing the weight propagation law.

### 3. UNIVERSAL-SPACE FORMULATION

#### 3.1 Appeal of the Universal Space

The rank-deficient parametric adjustment differs from most practical adjustments in one important aspect, the rank deficiency of the design matrix  $A$ . In tensor notations, the rank-deficient L.S. setup is represented by

$$dx^r = A_{\alpha}^r du'^{\alpha} + dx''^r, \quad (16)$$

where the first term on the right-hand side is equal to  $dx'^r$  in accordance with (12). Equation (16) closely resembles the situation depicted in [8] in all respects but the rank deficiency of the design tensor  $A_{\alpha}^r$ . In familiar adjustment notations, (16) would be transcribed as

$$L = AX - V,$$

where  $L$ ,  $X$ , and  $V$  are the vectors of constant terms, parametric corrections, and residuals, respectively. The vector  $L$  is formed as  $L^b - L^0$ , where  $L^b$  contains observations and  $L^0$  contains values of the observables consistent with an initial set of parameters. As can be gathered from Table 1, the geometry corresponding to a rank-deficient model yields the relations equivalent to singular normal equations in adjustment calculus:

$$a'_{\beta\alpha} du'^{\alpha} = du'_{\beta}; \quad (17a)$$

$$a'_{\beta\alpha} = A_{\beta}^s g_{sr} A_{\alpha}^r, \quad du'_{\beta} = A_{\beta}^s g_{sr} dx^r, \quad (17b,c)$$

where  $dx^r$ ,  $A_{\alpha}^r$ , and  $g_{sr}$  are given. Although all tensors in (17a-c) except  $du'^{\alpha}$  are known, the latter cannot be computed from (17a) without further stipulations, due to the singularity of  $a'_{\beta\alpha}$ , the necessary metric tensor.

Just as  $du'^{\alpha}$  represents the parametric corrections in tensor notations, the necessary associated metric tensor  $a'^{\alpha\beta}$  corresponds to the variance-covariance matrix of parameters. As Table 1 suggests, both these quantities can be expressed in theory with the aid of the associated metric tensor of the parametric space,  $a^{\alpha\beta}$ :

$$du'^{\alpha} = a'^{\alpha\beta} du'_{\beta} = a'^{\alpha\beta} du'_{\beta}, \quad (18a)$$

$$a'^{\alpha\beta} = a^{\alpha\gamma} a'_{\gamma\delta} a^{\delta\beta} = a^{\alpha\beta} - a^{\alpha\gamma} a''_{\gamma\delta} a^{\delta\beta}, \quad (18b)$$

where the second equality in (18b) is the consequence of  $a^{\alpha\beta} = a^{\alpha\gamma} a_{\gamma\delta} a^{\delta\beta}$ . The tensor  $a^{\alpha\beta}$  can be obtained from

$$a^{\alpha\beta} a_{\beta\gamma} = \delta_{\gamma}^{\alpha} \quad (19)$$

provided the metric tensor  $a_{\beta\alpha}$  is known.

Under the same assumption of known  $a_{\beta\alpha}$ , the tensors  $dx'^r$  and  $g'^{rs}$  are given by (15a,b), namely

$$dx'^r = A_{\alpha}^r du'^{\alpha} \quad (20a)$$

$$g'^{rs} = A_{\alpha}^r a^{\alpha\beta} A_{\beta}^s = A_{\alpha}^r a'^{\alpha\beta} A_{\beta}^s \quad (20b)$$

Parallel to (18b), Table 1 yields the tensor  $g'_{sr}$  as

$$g'_{sr} = g_{sp} g'^{pq} g_{qr} \quad (21)$$

which follows also from (15d) with (15e). The tensors  $dx'^r$ ,  $g'^{rs}$ , and  $g'_{sr}$  correspond to the adjusted observations in a linearized model, their variance-covariance matrix, and their weight matrix, respectively. As has been stated in the previous chapter, the pertinent functional-space contravariant tensors would be treated in complete analogy to  $dx'^r$  and  $g'^{rs}$  from (20a,b), except that  $dx'^r$ ,  $g'^{rs}$ , and  $A_{\alpha}^r$  would be attributed the symbol  $\sim$ . On the other hand, one cannot compute  $\tilde{g}'_{sr}$  in a functional-space version of (21) because the metric tensor in the functional space is unknown.

The foregoing represents a simple and plausible resolution of the rank-deficient model, hinging only on the availability of the metric tensor  $a_{\beta\alpha}$ . Unfortunately, this model offers no tensor relation containing  $a_{\beta\alpha}$ , and no indication of how the latter could be obtained. But it is clear that if the isomorphic geometrical setup could be "enlarged" into one where the parametric space constituted a proper model surface, the metric tensor for such a surface could be expressed following the simple approach of [B]. One is thus motivated to turn to the universal space in the role of an enlarged observational space, guided by the realization that the parametric space can indeed be considered as a new model surface embedded in the enlarged observational space.

### 3.2 Basic Universal-Space Setup

In pursuing the idea suggested in the previous section, we present a geometrical situation called "modified", where the model surface is extended to coincide with the parametric space. The latter is embedded in the universal space by construction. The modified surface, endowed with the coordinate system  $\{u^\alpha\}$ ,  $\alpha=1,2,\dots,u$ , is spanned by  $u$  orthonormal vectors  $\ell, j, \dots, t, \dots$ . And the universal space, endowed with the coordinate system  $\{X^R\}$ ,  $R=1,2,\dots,N$ , is spanned by  $N=u+n$  orthonormal vectors  $\ell, j, \dots, t, \dots, \nu, \dots$ . Suppose momentarily that the following three tensors are known:

$$dX^R = a\ell^R + bj^R + \dots + zt^R + \dots + q\nu^R + \dots, \quad (22a)$$

$$A_\alpha^R = \partial X^R / \partial u^\alpha = \ell^R \ell_\alpha + j^R j_\alpha + \dots + t^R t_\alpha + \dots, \quad (22b)$$

$$g^{RS} = \ell^R \ell^S + j^R j^S + \dots + t^R t^S + \dots + \nu^R \nu^S + \dots \quad (22c)$$

This case would then represent the situation discussed in [B], if one overlooks the conceptually unimportant addition of  $t, \dots$  to the original set of  $u'$  model-surface base vectors  $\ell, j, \dots$ .

A full-rank L.S. setup now corresponds to

$$dX^R = dX'^R + dX''^R, \quad (23a)$$

$$dX'^R = A_\alpha^R du^\alpha, \quad (23b)$$

where  $dX'$  and  $du$  are the same geometrical object expressed in different coordinate systems, represented by "a" units along the vector  $\ell$ , "b" units along the vector  $j, \dots$ , "z" units along the vector  $t, \dots$ . On the other hand,  $dX''$  is represented by "q" units along the vector  $\nu, \dots$ . Similar to [B], the relation

$$(A_\beta^S g_{SR} A_\alpha^R) du^\alpha = A_\beta^S g_{SR} dX^R, \quad (24)$$

where the tensor in parentheses is the metric tensor  $a_{\beta\alpha}$  of the modified model surface, is equivalent to (nonsingular) normal equations.

The metric tensor  $g_{SR}$  needed in (24) can be computed from the associated metric tensor  $g^{RS}$ . Similarly,  $a^{\alpha\beta}$  can be computed from  $a_{\beta\alpha}$ , giving rise to the solution  $du^\alpha = a^{\alpha\beta} du_\beta$ , where

$$du^\alpha = a\ell^\alpha + bj^\alpha + \dots + zt^\alpha + \dots,$$

$$a^{\alpha\beta} = \ell^\alpha \ell^\beta + j^\alpha j^\beta + \dots + t^\alpha t^\beta + \dots,$$

which already appeared in Table 1. The analytical form of the covariant tensors  $du_\beta$  and  $a_{\beta\alpha}$  taking part in (24) would be written in analogy to the above, with subscripts replacing the superscripts. Paralleling the demonstration in [8], the standard L.S. criterion is reflected by

$$dX''^S g_{SR} dX''^R = q^2 + \dots = \text{minimum}. \quad (25)$$

### 3.3 First Stipulation

The case presented above is of little practical use because none of the quantities  $dX^R$ ,  $A_\alpha^R$ , and  $g^{RS}$  is part of an actual geometrical setup. But upon using two stipulations, such a system can be developed into one featuring quantities that are either known from the rank-deficient model or can be chosen essentially arbitrarily. The first stipulation affects vectors' configuration by restricting  $dX$  to the original ( $n$ -dimensional) observational space, i.e., by enforcing  $z=\dots=0$ . This means that the orthonormal vectors  $t, \dots$  no longer play any role in describing  $dX$ ,  $dX'$ , or  $du$ . Since the vector  $dX'$  and its equivalent,  $du$ , are now restricted to the original ( $u'$ -dimensional) model surface, it is more convenient to replace the notation  $du$  by  $du'$  and write  $dX' \equiv du'$ .

Consistent with the first stipulation, (22a) becomes

$$dX^R = a\ell^R + bj^R + \dots + q\nu^R + \dots \quad (26a)$$

For the sake of completeness, we also list

$$dX'^R = a\ell^R + bj^R + \dots, \quad (26b)$$

$$dX''^R = q\nu^R + \dots \quad (26c)$$

The vector  $du'$  is expressed as in Table 1. With regard to the geometrical representation, equations (26a-c) and Table 1 indicate that the vectors  $dX$ ,  $dX' \equiv du'$ , and  $dX''$  are the same objects as  $dx$ ,  $dx' \equiv du'$ , and  $dx''$ , respectively. Thus, in considering Fig. 1, the vector denoted  $dx$  could be described by the symbols  $dX$ ,  $dx$ ; the vector denoted  $dx'$ ,  $du'$  could be described by the symbols



$dx'$ ,  $dx''$ ,  $du'$ ; and the vector denoted  $dx$  could be described by the symbols  $dx''$ ,  $dx'$ .

The first stipulation has not changed anything on tensors in (22b,c). And although it has altered the geometry of  $dx$  and  $dx'$ , it has not affected the form of (23a) as is readily apparent from (26a-c). Equations (23b) and (24) also remain the same, except that the notation  $du'$  now replaces  $du$ . With this change, equations (23a,b) are recapitulated for future reference as

$$dx^R = dx'^R + dx''^R, \quad (27a)$$

$$dx'^R = A_{\alpha}^R du'^{\alpha}. \quad (27b)$$

In the same vein, (24) is recapitulated in the form

$$a_{\beta\alpha} du'^{\alpha} = du'_{\beta}; \quad (28a)$$

$$a_{\beta\alpha} = A_{\beta}^S g_{SR} A_{\alpha}^R, \quad du'_{\beta} = A_{\beta}^S g_{SR} dx^R. \quad (28b,c)$$

### 3.4 Second Stipulation

The second stipulation pertains to the choice of the universal space coordinate system, and, as such, cannot be included in Fig. 1 or a similar geometrical illustration. This coordinate system is defined by

$$\{X^R\} = \{x^r, w^{\Lambda}\}, \quad (29)$$

where  $\{x^r\}$  and  $\{w^{\Lambda}\}$  represent coordinate systems in their own right. The latter are depicted in (5a) and (5d), and belong to the observational space and the extension surface, respectively. Since the observational space and the extension surface are orthocomplements of each other in the universal space, it follows in perfect analogy to the interim coordinate system of Chapter 2 that the universal space coordinate system is normal. In drawing on this analogy, we can state that the last  $u''$  universal-space components, contravariant and covariant, of vectors lying in the observational space are zero, as are the first  $n$  components of vectors lying in the extension surface. The remaining universal space components are identical to their counterparts formed in the respective subspaces. These properties are used extensively in the next three paragraphs.

The universal-space components of the orthonormal vectors lying in the observational space are

$$\ell^R = (\ell^r, 0) , \quad j^R = (j^r, 0) , \quad \dots , \quad \nu^R = (\nu^r, 0) , \quad \dots , \quad (30a)$$

$$\ell_R = (\ell_r, 0) , \quad j_R = (j_r, 0) , \quad \dots , \quad \nu_R = (\nu_r, 0) , \quad \dots , \quad (30b)$$

while the components of the orthonormal vectors lying in the extension surface are

$$t^R = (0, t^\Lambda) , \quad \dots , \quad (30c)$$

$$t_R = (0, t_\Lambda) , \quad \dots , \quad (30d)$$

With (30a-d), the unknown tensors of the universal-space setup can easily be separated into those given as a part of the original rank deficient model, and those that are unknown but can be chosen at will (subject to some general restrictions). For example, the components in (26a c) are seen to be

$$dx^R = (dx^r, 0) , \quad dx'^R = (dx'^r, 0) , \quad dx''^R = (dx''^r, 0) , \quad (31a,b,c)$$

which could also be written in the covariant version. The entire set  $dx^R$  is now known because the set  $dx^r$  represents the known components of the observational vector.

If we apply equations (30a,c) to (22b), we obtain

$$A_\alpha^R = (A_\alpha^r, \bar{Q}_\alpha^\Lambda) , \quad (32)$$

where both tensors on the right-hand side appeared in Table 1. If the same equations are further applied to (22c), it follows that

$$g^{RS} = (g^{rs}, k^{\Lambda\Omega}) , \quad (33)$$

where both tensors on the right-hand side also appeared in Table 1. Equation (33) can be related to the normal system described by (8a) and the text that followed. Similar to (8b), we also have

$$g_{SR} = (g_{sr}, k_{\Omega\Lambda}) . \quad (34)$$

Equations (32)-(34) contain tensors which are known from the rank deficient model, namely  $A_\alpha^r$ ,  $g^{rs}$ , and  $g_{sr}$ , as well as tensors which are as yet unknown, namely  $\bar{Q}_\alpha^\Lambda$ ,  $k^{\Lambda\Omega}$ , and  $k_{\Omega\Lambda}$ . We notice that with (31c) and (34), the quadratic form (25) becomes

$$dx''^s g_{sr} dx''^r = q^2 + \dots = \text{minimum} , \quad (35)$$

which corresponds to the standard L.S. criterion in the original formulation (full rank as well as rank-deficient). This confirms that the basic premise of the geometrical setup is kept intact by the universal-space approach.

Finally, using (31a-c) and (32), we can rewrite (27a,b) as

$$(dx^r, 0) = (A_\alpha^r, Q_\alpha^\Lambda) du'^\alpha + (dx''^r, 0) . \quad (36)$$

Similarly, using (31a), (32), and (34) in equations (28a-c), we obtain

$$a_{\beta\alpha} du'^\alpha = du'_\beta ; \quad (37a)$$

$$a_{\beta\alpha} = a'_{\beta\alpha} + a''_{\beta\alpha} , \quad a'_{\beta\alpha} = A_\beta^s g_{sr} A_\alpha^r , \quad a''_{\beta\alpha} = \bar{Q}_\beta^0 k_{\Omega\Lambda} \bar{Q}_\alpha^\Lambda ; \quad (37b,c,d)$$

$$du'_\beta = A_\beta^s g_{sr} dx^r . \quad (37e)$$

The above relations, consistent with Table 1, represent normal equations in the universal space approach. As their notations indicate, the necessary metric tensors  $a'_{\beta\alpha}$  and  $a''_{\beta\alpha}$  pertain respectively to the model surface and to the extension surface. The tensors  $a'_{\beta\alpha}$  and  $du'_\beta$  are known from the rank deficient formulation of normal equations, and are given explicitly in (17b,c), while the tensor  $a''_{\beta\alpha}$  is as yet unknown.

### 3.5 General Form of the Universal-Space Resolution

The foregoing development has illustrated that although the notion of normal coordinates plays a substantial role in the derivation of tensor equations suited to our needs, the coordinates themselves are neither needed nor known. Indeed, if tensor equations are established in one coordinate system, they are valid in any coordinates. As a fundamental feature of this chapter, such equations, formulated in the universal space endowed with the normal coordinate system  $\{X^R\}$ , allow us to transfer the desired tensors from the rank-deficient context to the familiar full-rank context. This advantage is best reflected by comparing equations (17a) and (37a), where the singular tensor  $a'_{\beta\alpha}$  in the former is replaced by the nonsingular tensor  $a_{\beta\alpha}$  in the latter. The formulation of the basic quantities  $du'^\alpha$  and  $a'^{\alpha\beta}$  then proceeds as in (18a) and (18b), respectively. And the formulation of  $dx'^r$ ,  $g'^{rs}$ ,  $g'_{sr}$ , and the remaining quantities of interest proceeds as outlined in (20a)-(21) and the text

that followed. We have thus witnessed how the universal space approach facilitates the resolution of the rank deficient model conceptually, through tensor equations. In the next step, we address the task of choosing the tensors as yet unknown in order to resolve this model numerically as well.

Equations (37b-d) show that  $a_{\beta\alpha}$  is obtained from  $a'_{\beta\alpha}$  by the addition of an essentially arbitrary tensor  $a''_{\beta\alpha}$ . The choice of the elements of  $a''_{\beta\alpha}$  is made through  $k_{\Omega\Lambda}$  and  $Q_{\alpha}^{\Lambda}$ . The tensor  $k_{\Omega\Lambda}$  must be positive definite (symmetric), but otherwise can be completely arbitrary. The tensor  $Q_{\alpha}^{\Lambda}$  must have the full rank in  $\Lambda$ , and, when joined to  $A_{\alpha}^{\Gamma}$ , must form a full rank tensor in  $\alpha$ , but otherwise can also be completely arbitrary. As the process leading from (25) to (35) attests, the choice of these two tensors has no bearing on the L.S. criterion.

Although the rank deficient setup is now solved in general, the numerical outcome for the basic tensors  $du'^{\alpha}$  and  $a'^{\alpha\beta}$  is non unique, due to the arbitrariness in  $a''_{\beta\alpha}$  propagated into  $a_{\beta\alpha}$ . The nature of this arbitrariness could be related to the coordinate system  $\{u^{\alpha}\}$  and its variations, but the present study, concerned with tensor relations and tensor components, has no need to link the latter to any coordinates explicitly. We observe, however, that the tensors  $dx'^{\Gamma}$ ,  $g'^{rs}$ , and  $g'_{sr}$  from (20a,b) and (21) are unique. A similar statement can be made with respect to the functional space version of (20a,b), whereas such a version is inconsequential for (21) due to the unknown metric tensor  $\bar{g}_{sr}$ . The uniqueness property is rooted in the fact that the components expressed in terms of  $\{u^{\alpha}\}$  are eliminated by tensor contractions.

We note that if the tensor  $Q_{\alpha}^{\Lambda}$  were given, the outcome for  $du'^{\alpha}$  and  $a'^{\alpha\beta}$  would also be unique, regardless of  $k_{\Omega\Lambda}$ . This stems from the fact that

$$a'_{\beta\alpha} \ell^{\alpha} = \ell_{\beta} \quad Q_{\alpha}^{\Lambda} \ell^{\alpha} = 0$$

and from similar identities for the parametric-space components of the remaining orthonormal vectors lying in the model surface,  $j, \dots$ . The first identity offers  $u'$  independent relations for  $\ell^{\alpha}$ ,  $\alpha=1,2,\dots,u$ , and the second identity provides additional  $u''$  independent relations. The component sets  $\ell^{\alpha}$ ,  $j^{\alpha}$ , are thus expressible through  $a'_{\beta\alpha}$  and  $Q_{\alpha}^{\Lambda}$ , with  $k_{\Omega\Lambda}$  playing no role at all. Implied in this demonstration is the requirement that  $(a'_{\beta\alpha}, Q_{\alpha}^{\Lambda})$  must have the full rank in  $\alpha$  which is satisfied due to the above condition for  $Q_{\alpha}^{\Lambda}$ .

In principle, with only  $a'_{\beta\alpha}$  and  $du'_\beta$  known, the outcome for  $a''_{\beta\alpha}$ ,  $a'_{\beta\alpha}$ ,  $a^{a\beta}$ , and  $a^{a\beta}$  is affected by both tensors  $k_{\Omega\Lambda}$  and  $Q^\Lambda_\alpha$ , the outcome for  $du'^a$  and  $a^{a\beta}$  is affected by  $Q^\Lambda_\alpha$  alone, and the outcome for  $dx'^r$ ,  $g'^{rs}$ , etc., discussed in the previous paragraph, is unique, affected by neither of the two tensors

We now summarize the key formulas from this chapter, transcribing them in matrix notations according to the simple convention introduced earlier (the indices are dropped and the symbols representing covariant tensors are attributed \*) We begin with the universal space setup (36), which now reads

$$\begin{bmatrix} dx' \\ 0 \end{bmatrix} = \begin{bmatrix} A \\ Q \end{bmatrix} du' + \begin{bmatrix} dx'' \\ 0 \end{bmatrix} \quad (38)$$

where the first matrix on the right hand side has the full column rank  $u$ . Since  $u$  is the smallest possible number of rows which, when augmenting  $A$ , can raise its rank to  $u$ , the second equation in (38) represents what is referred to in adjustment literature as minimal constraints,  $Q$  being the minimal constraint matrix. In practice, this matrix is often supplied or chosen beforehand. It is currently regarded as arbitrary, provided it fulfills the conditions

$$\text{rank}(Q) = u'', \quad \text{rank} \begin{bmatrix} A \\ Q \end{bmatrix} = u \quad (39a, b)$$

Clearly, (39b) could not hold true without (39a). Although the necessary condition (39a) need not be listed separately, it can serve in a first instance scrutiny of  $Q$ . The rank condition (39b) could equivalently be written with  $A^*$  replacing  $A$ .

The universal space approach treats (38) as a full rank adjustment model with the weight matrix  $\text{diag.}(g^*, k^*)$ . The latter is composed of the diagonal submatrices  $g^*$  and  $k^*$ , while the off diagonal submatrices are zero. The symbols  $g^*$  and  $k^*$  denote weight matrices in their own right, the first belonging to the observational space and the second belonging to the extension surface. The positive definite (symmetric) matrix  $g^*$  is the weight matrix of observations obtained as  $g^* = g^{-1}$ , where  $g$  is a given variance covariance matrix of observations. On the other hand, the symmetric matrix  $k^*$  is arbitrary, subject only to the condition that it must be positive definite. The geometrical

considerations above have indicated that although  $k^*$  affects  $a^{a\beta}$ ,  $a^*$ ,  $a$ , and  $a''$ , this matrix has no bearing on the determination of  $du'$  and  $a'$  (in addition to

$dx'$ ,  $g'$ , etc.). This is consistent with the statement on page 17 of [Pope, 1973], that "minimally constrained solutions do not depend on  $\Sigma$  associated with the minimal constraint", where  $\Sigma$  is our  $(k^*)^{-1} = k$ .

In forming normal equations from (38), one obtains formulas paralleling (37a-e):

$$a^* du' = du^{*'}; \quad (40a)$$

$$a^* = a^{*'} + a^{*''}, \quad a^{*'} = A^T g^* A, \quad a^{*''} = \bar{Q}^T k^* \bar{Q}; \quad (40b,c,d)$$

$$du^{*'} = A^T g^* dx. \quad (40e)$$

Here the quantities known from the rank-deficient L.S. setup are  $A$  (the design matrix),  $g^*$  (the weight matrix of observations), and  $dx$  (the vector of observations in a linearized model), which give rise to  $a^{*'}$  (the singular matrix of normal equations) and  $du^{*'}$  (the right-hand side of normal equations). According to (18a,b),  $du'$  and its variance covariance matrix  $a'$  are computed from

$$du' = a du^{*'} = a' du^{*'}, \quad (41a)$$

$$a' = a a^{*'} a = a - a a^{*''} a, \quad (41b)$$

where, in view of (19),

$$a = (a^*)^{-1}. \quad (42)$$

The second equality in (41b) uses the fact that  $a'' = a a^{*''} a$ . The second equalities in both of (41a,b) can serve for verification purposes.

With regard to the vector  $dx'$  containing the adjusted observations in a linearized model, and to its variance covariance matrix  $g'$ , from (20a,b) we transcribe

$$dx' = A du', \quad (43a)$$

$$g' = A a A^T = A a' A^T. \quad (43b)$$

The second equality in (43b) can again serve for verifications. If needed, the weight matrix  $g^{*'}$  associated with  $dx'$  can be transcribed from (21) as

$$g^{*'} = g^* g' g^* \quad (44)$$

Although the matrix  $a^{**}$  in (40d) and thus also  $a^*$ ,  $a$ , and  $a''$  are non-unique due to admissible variations in  $\bar{Q}$  and  $k^*$ , and although the matrix  $a'$  as well as the vector  $du'$  are non-unique due to admissible variations in  $\bar{Q}$  alone, the quantities  $dx'$ ,  $g'$ , and  $g^{**}$  above are unique. Equations (43a,b) could also be used in conjunction with linear functions of  $du'$ , in which case  $A$ ,  $dx'$ , and  $g'$  would be attributed the symbol  $\sim$ . The vector  $\tilde{dx}'$  and its variance-covariance matrix  $\tilde{g}'$  would then be unique as well. On the other hand, the weight matrix  $\tilde{g}^{**}$  could not be computed in analogy to (44) because  $\tilde{g}^*$  is unknown. Since the solution  $du'$  and its variance-covariance matrix  $a'$  become unique if  $\bar{Q}$  is specified numerically, the latter will be subject to further discussion.

## 4. MINIMAL-CONSTRAINT FORMULATION

### 4.1 Geometrical Background

The treatment of the rank-deficient adjustment in the preceding chapter has resulted in an extended full-rank formulation, such as presented in (38) and beyond. Here we describe another approach, where the rank-deficient adjustment is addressed via minimal constraints. We begin with a geometrical setup in the universal space, comprising two sets of equations. The first set depicts the rank deficient L.S. setup (16), or, equivalently, the tensor form of singular normal equations appearing in (17a):

$$a'_{\beta\alpha} du'^{\alpha} = du'_{\beta} \quad (45a)$$

And the second set reads

$$\bar{Q}^{\Lambda}_{\alpha} du'^{\alpha} = 0 \quad (45b)$$

which, in itself, is an identity as indicated by Table 1. But when considered in conjunction with (45a), it ensures that  $du'^{\alpha}$  cannot be substituted for by  $du^{\alpha} = du'^{\alpha} + du''^{\alpha}$  with a nonzero set  $du''^{\alpha}$ . In terms of geometrical objects, it ensures that the model vector  $du'$  cannot be substituted for by a general vector  $du$  from the parametric space as could be done in (45a) alone, i.e., that  $du'' \neq 0$ , where  $du''$  is a vector lying in the extension surface.

In elaborating on this assertion, we express the components of a vector lying in the extension surface in two coordinate systems. We use the system  $\{u^{\alpha}\}$  from (5b), in which case the vector is symbolized by  $du''$ , and the system  $\{w^{\Lambda}\}$  from (5d), in which case it is symbolized by  $dw$ . The geometrical object  $du'' \equiv dw$  is described analytically in Table 1. This table enables us to relate component sets of  $du'' \equiv dw$  to each other through the  $\bar{Q}$ -tensor by

$$dw^{\Lambda} = \bar{Q}^{\Lambda}_{\alpha} du''^{\alpha} \quad , \quad du''_{\beta} = \bar{Q}^{\Omega}_{\beta} dw_{\Omega} \quad (46a,b)$$

If  $du'' \equiv dw$  is a zero vector, all its components are zero in any coordinates, and vice versa. Suppose now that the component set  $du'^{\alpha}$  in (45a,b) is substituted for by  $du^{\alpha} = du'^{\alpha} + du''^{\alpha}$ . The new equation (45b) then becomes

$$0 = \bar{Q}^{\Lambda}_{\alpha} du'^{\alpha} + \bar{Q}^{\Lambda}_{\alpha} du''^{\alpha} = \bar{Q}^{\Lambda}_{\alpha} du^{\alpha} = dw^{\Lambda} \quad ,$$



where use has been made of (46a). But this means that  $du'' dw$  is restricted to zero. We can thus conclude that (45b) used in conjunction with any relation containing  $du'^\alpha$  ensures that no general set  $du''^\alpha$  is allowed to replace  $du'^\alpha$ .

#### 4.2 General Form of the Minimal Constraint Resolution

We have just seen that no components  $du''^\alpha$  other than zero are allowed in the solution of (45a,b). Thus, since  $du''$  is a zero vector, one can use (46b) with  $du''_\beta = 0$  and write

$$\bar{Q}^\Omega_\beta dw_\Omega = 0 .$$

These zero components can be freely added to (45a), which, when combined with (45b), gives rise to the system

$$a'_{\beta\alpha} du'^\alpha + \bar{Q}^\Omega_\beta dw_\Omega = du'_\beta , \quad (47a)$$

$$\bar{Q}^\Lambda_\alpha du'^\alpha = 0 . \quad (47b)$$

Equations (47a,b) represent the standard formulation of normal equations with (absolute) constraints in adjustment calculus, where the new notation  $dw_\Omega$  corresponds to the Lagrange multipliers. These multipliers are thus given a clearcut geometrical interpretation illustrating why they must be zero in a rank-deficient adjustment with minimal constraints.

In working with matrix notations from this point on, we transcribe the system (47a,b) as

$$\begin{bmatrix} a^{*'} & \bar{Q}^T \\ Q & 0 \end{bmatrix} \begin{bmatrix} du' \\ dw^* \end{bmatrix} = \begin{bmatrix} du^{*'} \\ 0 \end{bmatrix} . \quad (48)$$

An analytical inversion of the regular (symmetric) matrix in (48) necessitates the following matrix identities transcribed from their tensor counterparts:

$$\begin{aligned} a'_{\alpha\beta} a'^{\beta\gamma} + Q^\Omega_\alpha \bar{A}^\gamma_\Omega &= \delta^\gamma_\alpha & \dots & \dots & a^{*'} a' + \bar{Q}^T A^T &= I , \\ a'_{\alpha\beta} \bar{A}^\beta_\psi &= 0 & \dots & \dots & a^{*'} \bar{A} &= 0 , \\ \bar{Q}^\Lambda_\beta a'^{\beta\gamma} &= 0 & \dots & \dots & \bar{Q} a' &= 0 , \\ \bar{Q}^\Lambda_\beta \bar{A}^\beta_\psi &= \delta^\Lambda_\psi & \dots & \dots & \bar{Q} \bar{A} &= I . \end{aligned}$$

The matrix relations presented above yield

$$\begin{bmatrix} du' \\ dw^* \end{bmatrix} = \begin{bmatrix} a' & \bar{A} \\ \bar{A}^T & 0 \end{bmatrix} \begin{bmatrix} du^{*'} \\ 0 \end{bmatrix} \quad (49)$$

That the matrix in (49) is the inverse of the matrix in (48) is demonstrated upon forming their product in either order and obtaining I. This constitutes an easy proof, in retrospect, that the matrix in (48) is regular. Equation (49) results in

$$du' = a' du^{*'} , \quad dw^* = 0 , \quad (50a,b)$$

where (50a) is the second equality in (41a), and (50b) corresponds to the identity  $\bar{A}_{\alpha}^{\alpha} du'_{\alpha} = 0$  implied by Table 1.

According to the adjustment theory, the variance-covariance matrix of the parametric solution is the leading submatrix in (49). This can be regarded as an independent confirmation that the necessary associated metric tensor  $a'^{\alpha\beta}$  indeed corresponds to the variance-covariance matrix sought by the adjustment. Further agreement with the theory is evidenced by the results (50a,b) themselves, as well as by the zero diagonal submatrix in (49). One notices that (49) furnishes  $a'$  without proceeding through an inversion of the positive-definite matrix  $a^*$  of dimensions  $(u \times u)$  as was the case in Chapter 3. However, the current procedure requires an inversion of the matrix in (48), whose dimensions are  $[(u+u'') \times (u+u'')]$ , and which is not positive-definite. With regard to  $dx'$ ,  $g'$ ,  $g^*$ , and, eventually,  $\tilde{dx}'$  and  $\tilde{g}'$ , equations (43a), (43b) without the middle equality, and (44), as well as the text that followed, apply perfectly well also in the present situation. Similar to the closing statement in Chapter 3, the numerical outcome for  $du'$  and  $a'$  depends on the explicit form of  $\bar{Q}$  yet to be considered.

## 5. ANALYTICAL FORMULATION

### 5.1 System of Orthonormal Vector Components

A complete description of a general object in parametric-space components, contravariant and covariant, is related to the components of all the orthonormal vectors  $\ell, j, \dots, t, \dots$ . Since  $\ell_\alpha, j_\alpha, \dots$  can be determined from the known tensor  $a'_{\beta\alpha}$ , it follows that  $t_\alpha, \dots$ , or an equivalent set of  $u^\alpha x_u$  components, must be chosen in some manner. The remaining components can then be found from

$$\ell^\alpha \ell_\beta + j^\alpha j_\beta + \dots + t^\alpha t_\beta + \dots = \delta^\alpha_\beta, \quad (51a)$$

or, equivalently, from the identities

$$\ell_\alpha \ell^\alpha = 1, \quad \ell_\alpha j^\alpha = 0, \dots \quad (51b)$$

As its analytical form suggests, the tensor  $\bar{Q}^\Lambda_\alpha$  is closely linked to the components of  $t_\alpha, \dots$ . Accordingly, a satisfactory description of this tensor can be made only after an analysis of interrelationships among the parametric-space components of  $\ell, j, \dots, t, \dots$ .

A collection of the above parametric-space components expressed in a given coordinate system  $\{u^\alpha\}$  is called "system of orthonormal vector components", or simply "system". Any such system must fulfill (51a,b). If  $\{u^\alpha\}$  changes, the system also changes (i.e., some or all of the covariant and contravariant components of  $\ell, j, \dots, t, \dots$  change), but the formulas (51a,b) must again be satisfied. In this study, one such system will be considered initial, and others will be considered its variants. However, the underlying  $\{u^\alpha\}$  and its variants are of no interest here.

The analysis is facilitated if the vector components are grouped in matrices in analogy to (13b,c). This does not detract from the geometrical nature of the present development, and does not constitute a mixed algebraic-geometrical approach. It is merely a convention, where the geometrical quality of the pertinent matrices is kept in focus through a matrix equivalent of (51a,b). The components of  $\ell, j, \dots, t, \dots$  are grouped as follows:

$$L^* = [[\ell_\alpha][j_\alpha] \dots], \quad L = [[\ell^\alpha][j^\alpha] \dots], \quad (52a,b)$$

$$T^* = [[t_\alpha] \dots], \quad T = [[t^\alpha] \dots], \quad (52c,d)$$

where  $\{l_\alpha\}$ , etc., represent column vectors. The matrices  $L^*$  and  $L$  have the dimensions  $(u \times u')$ , while  $T^*$  and  $T$  have the dimensions  $(u \times u'')$ . The grouping seen in (52a) already appeared as (13c). From their construction, it is clear that all four matrices above have the full column rank, which is  $u'$  for  $L^*$  and  $L$ , and  $u''$  for  $T^*$  and  $T$ .

The term "system" describing the collection of vector components can be used interchangeably in reference to the four matrices above, which assemble and arrange these components in a prescribed order. In view of (51a,b), any such system must fulfill the following criteria:

$$LL^{*T} + TT^{*T} = I, \quad (53)$$

$$L^{*T}L = I, \quad L^{*T}T = 0, \quad T^{*T}L = 0, \quad T^{*T}T = I. \quad (54a,b,c,d)$$

Equation (53) and the set (54a-d) are equivalent, expressing the conditions  $CD=I$  and  $DC=I$ , respectively, where the regular matrices  $C$  and  $D$  are formed as  $C=[L \ T]$  and  $D=[L^* \ T^*]^T$ . In either case, we have

$$[L \ T] = \{[L^* \ T^*]^T\}^{-1}. \quad (55)$$

In analogy to a previous statement, one system fulfilling (53) can be considered initial, and others can be considered its variants. Each of the latter must again satisfy the condition (53) or, equivalently, (54a-d). All possible variants of an initial systems (including the latter itself) are said to constitute a family of systems.

Of the four matrices  $L^*$ ,  $L$ ,  $T^*$ , and  $T$  forming a system, the matrix  $L^*$  is considered to be known since it can be obtained, for example, by the Choleski algorithm for positive semi-definite matrices. This algorithm is applied below to the matrix of normal equations,  $a^*$ , which can be transcribed from Table 1 as

$$a^* = L^*L^{*T}. \quad (56)$$

Without any loss of generality, the  $(u' \times u')$  leading submatrix of  $a^*$ , denoted  $N_{11}$ , can be assumed positive-definite. In practice, this is true either a priori, or can be achieved upon reordering the parameters. The other submatrices of  $a^*$  are, clockwise,  $N_{12}$ ,  $N_{22}$ , and  $N_{21}$ , where  $N_{21} = N_{12}^T$ .

Consistent with the partition of  $a^*$ , the matrices  $L^*$ ,  $L$ ,  $T^*$ , and  $T$  are partitioned into two submatrices each. The first submatrix contains  $u'$  rows and is attributed a prime, while the second submatrix contains the remaining  $u''$  rows and is attributed a double prime. The partitioned matrices are presented as

$$L^* = \begin{bmatrix} L^{*'} \\ L^{*''} \end{bmatrix}, \quad L = \begin{bmatrix} L' \\ L'' \end{bmatrix}, \quad T^* = \begin{bmatrix} T^{*'} \\ T^{*''} \end{bmatrix}, \quad T = \begin{bmatrix} T' \\ T'' \end{bmatrix}.$$

Due to the above stipulation for  $N_{11}$ , the submatrix  $L^{*'}$  must be regular. Since

$$N_{11} = L^{*'} L^{*'}{}^T,$$

it follows that  $L^{*'}{}^T$  can be determined by the familiar Choleski algorithm for positive definite matrices, which assigns zero values to  $u'(u'-1)/2$  arbitrary elements and groups them below the main diagonal. The submatrix  $L^{*'}{}^T$  is thus upper-triangular and  $L^{*'}$  is lower-triangular. From the submatrix  $N_{12}$  of  $a^*$ , one can determine  $L^{*''}$  through

$$L^{*''}{}^T = (L^{*'}{}^T)^{-1} N_{12}.$$

The submatrix  $N_{22}$  of  $a^*$  does not lead to any new relations; but utilizing the above two equations, we confirm that  $N_{22} = N_{21} N_{11}^{-1} N_{12}$  as it should.

Clearly, the "Choleski choice" for arbitrary elements of  $L^{*'}$  is neither a theoretical nor a practical necessity. In fact, there exists an infinite number of other acceptable choices, and they will be discussed in principle later. But for the time being,  $L^{*'}$  and thus the resulting matrix  $L^*$  represent fixed entities in our system. It follows from the above that

$$L^* = [I \quad R]{}^T L^{*'}, \quad (57)$$

where

$$R = (L^{*'}{}^T)^{-1} L^{*''}{}^T = N_{11}^{-1} N_{12}. \quad (58)$$

The matrix  $R$  of dimensions  $(u' \times u'')$  is known from  $a^*$ , and is fixed throughout.

So far, little has been said with regard to the matrices  $L$ ,  $T^*$ , and  $T$ , except that they must conform to (53) or (54a-d). The matrix  $L$  is especially important because it serves in forming  $a'$ , transcribed from Table 1 as

$$a' = LL^T, \quad (59)$$

and in forming  $du'$ , similarly transcribed as

$$du' = Ly, \quad (60)$$

where  $y$  is a column vector of  $u'$  elements, containing the constants  $a, b, \dots$ . The values in  $a'$  and  $du'$  depend on a particular choice of  $L$  from the family of systems, but the constants  $a, b, \dots$  are tensor invariants, expressed from Table 1 as

$$a = l^\alpha du'_\alpha, \quad b = j^\alpha du'_\alpha, \dots$$

In matrix notations, the unique vector  $y$  is thus given by

$$y = L^T du^{*'} = L^T A^T g^* dx. \quad (61)$$

Accordingly, equation (60) can also be written in the form

$$du' = LL^T du^{*'} = a' du^{*'}, \quad (60')$$

which appeared already in (41a) and (50a).

The resolution of a system can proceed along different avenues. For example, if the  $uxu''$  elements of the matrix  $T^*$  were given and fixed, both  $L$  and  $T$  would be determined with the final validity by (55). This illustrates the fact that in addition to the fixed matrix  $L^*$ , the determination of a system requires the knowledge, or choice, of  $uxu''$  independent elements. We shall proceed by dividing  $uxu''$  elements subject to choice into two separate groups of  $u''xu''$  and  $u'xu''$  elements, respectively. Since only the second group will play a role in the determination of  $L$  and thus of  $a'$  and  $du'$  from (59) and (60), the analysis based on the properties of  $a'$  and/or  $du'$  will be greatly facilitated.

## 5.2 Family of Systems

In order to describe a family of systems, we shall choose one member of the family as initial and then consider its variants as has been suggested in the preceding section. However, simultaneous variations will apply only to the matrices  $L$  and  $T^*$ . Whereas the matrix  $L^*$  is considered fixed from the beginning, the matrix  $T$  can be fixed in one "loop" of a nested approach leading to a sub-family of systems. This process can be imagined repeated with all allowable matrices  $T$ , eventually describing the entire family of systems consisting of the fixed matrix  $L^*$  and of the matrix families of  $T$ ,  $L$ , and  $T^*$ . The family of  $L$  will be shown independent of the variations in  $T$ , which will enable us to accomplish the analysis dealing with  $du'$  and  $a'$  without the nesting process. The matrix  $T$  is seen below to become fixed as soon as  $u \times u$  of its elements grouped in  $T''$  are chosen and held fixed. These elements comprise, in fact, the first group of elements subject to choice alluded to at the close of Section 5.1.

To determine a general matrix  $T$ , we use (54b) with partitioned  $L^*$  and  $T$ , where the former has been presented explicitly in (57). This yields  $T' = -RT''$ , and thus

$$T = [-R^T \quad I]^T T'' . \quad (62)$$

The submatrix  $T''$  must be regular due to the full column rank of  $T$ , but otherwise can be arbitrary. Its elements, and thus also the entire matrix  $T$ , are now considered fixed.

The symbols  $L$  and  $T^*$  henceforth refer to the respective families of matrices, while their initial choices are underlined. We thus have

$$L = \underline{L} + \Delta L , \quad (63)$$

$$T^* = \underline{T}^* + \Delta T^* , \quad (64)$$

where  $\Delta L$  and  $\Delta T^*$  symbolize the variants of the initial choices. The complete family of  $T^*$  can be described only after all allowable submatrices  $T''$  in (62) have been accounted for, whereas the family of  $L$  is independent of such variations. This will become apparent upon considering the basic conditions (54a-d), which must be fulfilled by the initial system as well as any of its variants.

Particularly simple and useful choices for  $\underline{L}$  and  $\underline{T}^*$  can be made in terms of the fixed matrices  $L^*$  and  $T$ , respectively, as follows:

$$\underline{L} = L^* (L^{*T} L^*)^{-1} , \quad (65)$$

$$\underline{T}^* = T (T^T T)^{-1} . \quad (66)$$

It is readily confirmed that with these initial matrices the conditions (54a-d) are satisfied. A valid initial system has thus been established. As will be explained, this system can also be called canonical due to the advantage offered by the form of  $\underline{L}$  and  $\underline{T}^*$  above.

We next proceed to determine the family of systems by applying the conditions (54a,c,d); the condition (54b) is fulfilled by (62) with any regular submatrix  $T''$  and need not be mentioned again. We shall formulate these conditions with (63) and (64) for  $L$  and  $T^*$ , utilizing the fact that they are already fulfilled with  $\underline{L}$  and  $\underline{T}^*$ . From (54a,c,d) we obtain, respectively,

$$L^{*T} \Delta L = 0 , \quad (67a)$$

$$\underline{T}^{*T} \Delta L + \Delta T^{*T} \underline{L} + \Delta T^{*T} \Delta L = 0 , \quad (67b)$$

$$\Delta T^{*T} T = 0 . \quad (67c)$$

The matrices  $\Delta L$  and  $\Delta T^*$  are partitioned in the same fashion as their counterparts  $L$  and  $T^*$  presented past equation (56). Upon considering (57), equation (67a) gives

$$\Delta L = [-R^T \quad I]^T \Delta L'' \equiv T(T'')^{-1} \Delta L'' , \quad (68)$$

where the last formulation follows from (62). On the other hand, (67c) results in  $\Delta T^* = L^* (L^{*T})^{-1} \Delta T^{*T}$ , where the partitioned form of  $L^*$  from (57) as well as the equality  $T' = -RT''$  from (62) have been taken into account. Since (67b) leads to  $\Delta T^{*T} = -L^{*T} \Delta L'' (T''^T)^{-1}$ , we obtain

$$\Delta T^* = -L^* \Delta L''^T (T''^T)^{-1} . \quad (69)$$

From (68), including the last formulation where  $T(T'')^{-1}$  is fixed throughout, we observe that  $\Delta L$  depends only on the  $u' \times u''$  elements of  $\Delta L''$ , which constitute the second group of elements subject to choice as mentioned at the close of Section 5.1. It thus follows from (65) and (68) that  $L$  varies only with the completely arbitrary submatrix  $\Delta L''$ , and is independent of  $T''$ . By contrast,  $T^*$



varies with both  $\Delta L''$  and  $T''$ . This is evidenced by (66) with (62) showing the dependence of  $\underline{T}^*$  on  $T''$ , and by (69) showing the dependence of  $\Delta T^*$  on  $T''$  as well as on  $\Delta L''$ .

We notice that (67a) alone leads to other useful identities. For example, due to the decomposition  $A = FL^{*T}$  in (13a), equation (67a) implies that

$$A\Delta L = 0 . \quad (70)$$

The latter part of Section 2.4 indicates that a relation similar to (13a) can be written in conjunction with the functional space, in which case  $A$  and  $F$  are attributed the symbol  $\tilde{\phantom{x}}$ . But since  $L^{*T}$  remains unchanged, (67a) also yields

$$\tilde{A}\Delta L = 0 . \quad (70')$$

The identity (70) enables us to confirm, via the matrix family of  $L$ , the uniqueness of  $y$  previously demonstrated via geometry. In particular, if we substitute  $L$  from (63) into the formula for  $y$  in (61), equation (70) shows that the term containing  $\Delta L$  is zero, so that

$$y = \underline{L}^T du^{*'} = \underline{L}^T A^T g^* dx . \quad (71)$$

Thus, no matter which member of the family  $L$  is used in the computation,  $y$  is unique and is expressed with advantage through  $\underline{L}$ .

The usefulness of the initial system adopted in this study can best be illustrated by means of the additional identities below. Due to  $L^{*T}T=0$  and the definition of  $\underline{L}$  and  $\underline{T}^*$ , it follows that

$$\underline{L}^T T = 0 , \quad \underline{T}^{*T} L^* = 0 . \quad (72a,b)$$

These equations are valid with any  $T''$ , but strictly with  $\underline{L}$  and  $\underline{T}^*$  (i.e.,  $\Delta L'' = 0$ ). Furthermore, in consulting (68) and (69), we deduce that

$$\underline{L}^T \Delta L = 0 , \quad \underline{T}^{*T} \Delta T = 0 , \quad (73a,b)$$

which are seen to hold true independent of  $\Delta L''$  and  $T''$ . The property (73a) will greatly facilitate the analysis of a family of expressions such as  $\underline{L}^T L$ , reducing it to the initial product  $\underline{L}^T \underline{L}$  plus the family of  $\Delta L^T \Delta L$ . The vanishing of cross products such as seen above is a trademark of canonical systems in various mathematical problems. In this sense, the special initial system used in the present analysis is canonical.

Next, we develop an explicit form for the families of  $L$  and  $T^*$ , similar to the expressions (57) and (62) for  $L^*$  and  $T$ . We begin with  $\underline{L}$ , whose formation in (65) entails an inversion of two positive definite matrices of dimensions  $(u' \times u')$ , the first being  $N_{11}$  needed in the computation of  $R$  and subsequently  $L^*$ , and the second being  $L^{*T} L^*$ . However, the latter can be replaced by an inversion of another positive definite matrix whose dimensions are  $(u'' \times u'')$ . Since in practice  $u'' < u'$ , this avenue holds an economical advantage. We now formulate  $\underline{L}$  from (65) in conjunction with (57):

$$\underline{L} = [I \ R]^T (I + RR^T)^{-1} (L^{*T})^{-1}, \quad (74)$$

where the positive definite matrix  $I + RR^T$  has the dimensions  $(u' \times u')$ . To develop an equivalent but more useful expression, we first write the identity

$$R^T = (I + R^T R)^{-1} R^T (I + RR^T), \quad (75a)$$

Upon premultiplying (75a) by  $R$ , adding  $I$  to both sides, and postmultiplying the new equation by  $(I + RR^T)^{-1}$  one obtains (75b) below. And upon postmultiplying (75a) by  $(I + RR^T)^{-1}$  one obtains (75c). The two generated identities read

$$(I + RR^T)^{-1} = I - R(I + R^T R)^{-1} R^T, \quad (75b)$$

$$R^T (I + RR^T)^{-1} = (I + R^T R)^{-1} R^T. \quad (75c)$$

We now introduce the notation  $H$  for the following matrix expression, needed in reformulating  $\underline{L}$  as well as  $\underline{T}^*$ :

$$H = [-R^T \ I]^T (I + R^T R)^{-1}, \quad (76)$$

where the positive-definite matrix  $I + R^T R$  has the dimensions  $(u'' \times u'')$  as stipulated. With the identities (75b,c), the matrix  $\underline{L}$  from (74) becomes

$$\underline{L} = \{[I \ 0]^T + HR^T\} (L^{*T})^{-1}. \quad (77)$$

The family of matrices  $\Delta L$  has been presented in (68) and need not be repeated. The new explicit formulation of the family of  $L$  then follows from (63) as

$$L = \{[I \ 0]^T + HR^T\} (L^{*T})^{-1} + [-R^T \ I]^T \Delta L, \quad (78)$$

where only  $\Delta L$  is variable. We reiterate that the  $u' \times u''$  elements of this submatrix are completely arbitrary.

With regard to  $\underline{I}^*$ , equations (62) and (66) yield

$$\underline{I}^* = H(T''^T)^{-1} \quad (79)$$

The family of matrices  $AT^*$  has been presented in (69). The new explicit formulation of the family of  $T^*$  then follows from (64) as

$$T^* = (H - L^* \Delta L''^T)(T''^T)^{-1} \quad (80)$$

featuring not only  $\Delta L''$  but also  $T''$  as variable. We reiterate that the  $u'' \times u''$  elements of the latter are arbitrary, subject only to the restriction that  $T''$  must be regular.

Equation (80) illustrates how the nested approach can be used in theory to describe the family of  $T^*$ . In a natural sequence, one first chooses a regular matrix  $T''$  and holds it fixed while varying  $\Delta L''$  over its range. The second and subsequent steps differ from the above only in a changed  $T''$ , until the latter has covered its entire range. However, to select a unique system of the family, one only needs to choose one submatrix  $T''$  and one submatrix  $\Delta L''$ , and use them in (62) giving  $T$  in terms of  $T''$ , in (78) giving  $L$  in terms of  $\Delta L''$ , and in (80) giving  $T^*$  in terms of both  $T''$  and  $\Delta L''$ .

In considering that the goals of the present analysis are intimately linked to the characteristics of  $du'$  and  $a'$ , it is expedient to concentrate on a subfamily of systems corresponding to a desirable sub-family of  $L$ . The hierarchy in the treatment of elements subject to choice can then be altered accordingly, based on a selected submatrix  $\Delta L''$  rather than  $T''$ . This submatrix yields a unique matrix  $L$  from (78), while the matrix  $T$  from (62) and the matrix  $T^*$  from (80) are non unique, depending on  $T''$  which is now variable. Clearly, each pair of  $T$  and  $T^*$  must use the same  $T''$ . At this stage not only  $L$ , but also  $du'$  and  $a'$  remain unique, unaffected by the variability in the elements of  $T''$ . But regardless of the hierarchy in the variable elements, for every one choice of  $\Delta L''$  and  $T''$ , the conditions (54a-d) are satisfied and the four matrices  $L^*$ ,  $L$ ,  $T^*$ , and  $T$  contain a valid set of parametric-space components of the orthonormal vectors  $l, j, \dots, t, \dots$

### 5.3 Families of Adjustment Quantities

As has been indicated in the latter part of Section 5.1, the matrix family of  $L$  plays an essential role in the determination of the adjustment quantities  $a'$  and  $du'$ . The description of  $L$  via the initial choice  $\underline{L}$  and its variants  $\Delta L$  leads to the description of these quantities along similar lines. A subsequent analysis is greatly facilitated by the canonical property of the initial choice. However, the adjustment quantities such as  $dx'$  and  $g'$  are independent of the parametric space components, which have been eliminated by tensor contractions. As geometrical considerations have already indicated, these quantities are unique and must therefore be obtainable with any choice of  $L$ . This will be confirmed by showing that the terms containing  $\Delta L$  vanish, similar to the discussion concerned with  $y$  in the preceding section.

The family of  $a'$  is described by (59) together with (63) in a straightforward fashion as

$$a' = \underline{a}' + \Delta L \Delta L^T + \underline{L} \Delta L^T + \Delta L \underline{L}^T, \quad (81)$$

where

$$\underline{a}' = \underline{L} \underline{L}^T. \quad (82)$$

Equation (81) cannot be simplified because in general  $\underline{L} \Delta L^T \neq 0$ . However, the trace of the latter is equal to the trace of  $\underline{L}^T \Delta L$ , which is zero by virtue of the canonical initial system as evidenced by (73a). In terms of traces, we thus have

$$\text{Tr}(a') = \text{Tr}(\underline{a}') + \text{Tr}(\Delta L \Delta L^T). \quad (83)$$

The first term on the right hand side can also be written as  $\text{Tr}(\underline{L}^T \underline{L})$ , which is  $\text{Tr}[(L^*{}^T L^*)^{-1}]$ , and the second term can also be written as  $\text{Tr}(\Delta L^T \Delta L)$ .

The family of  $du'$  is described by (60) together with (63) as

$$du' = \underline{du}' + \Delta L y, \quad (84)$$

where

$$\underline{du}' = \underline{L} y. \quad (85)$$

The most convenient form of the column vector  $y$  has been presented in (71).

Equation (85) can thus be written as

$$\underline{du}' = \underline{L}\underline{L}^T du^{*'} = \underline{a}' du^{*'} \quad (85')$$

which corresponds to the canonical member of (60'). Finally, in describing the family of  $du'^T du'$ , we again take advantage of (73a) and obtain

$$du'^T du' = \underline{du}'^T \underline{du}' + (\Delta Ly)^T (\Delta Ly) \quad (86)$$

We now turn to the adjustment quantities independent of the parametric-space components and confirm their uniqueness. In particular, using (70) together with (84) we deduce that

$$dx' = Adu' = A\underline{du}' \quad (87)$$

Similarly, (70) in conjunction with (81) yields

$$g' = Aa'A^T = A\underline{a}'A^T \quad (88)$$

Thus, no matter which members of the families of  $du'$  and  $a'$  are used in the computation,  $dx'$  and  $g'$  are unique; they are expressed with advantage through the canonical members  $\underline{du}'$  and  $\underline{a}'$ , respectively. We can write relations similar to (87) and (88) in conjunction with the functional space, in which case  $A$ ,  $dx'$ , and  $g'$  are attributed the symbol  $\sim$ . The uniqueness of parametric functions and their variance-covariance matrix is then confirmed via (70').

#### 5.4 Minimum-Trace and Minimum-Norm Criteria

We first examine the conditions resulting in a minimum trace of the variance-covariance matrix of the parameters, i.e., in  $\text{Tr}(a') = \text{minimum}$ . The groundwork for this task has been laid in the preceding section, where (83) expresses the family of  $\text{Tr}(a')$  in terms of the fixed part  $\text{Tr}(\underline{a}')$  and the variable part  $\text{Tr}(\Delta L \Delta L^T)$ . Since the latter equals the sum of squares of all the elements in  $\Delta L$ , the necessary and sufficient condition for (83) to achieve a minimum is

$$\Delta L = 0 \quad (89)$$

Accordingly, the family of  $a'$  in (81) reduces to  $\underline{a}'$  given by (82), and the family of  $du'$  in (84) reduces to  $\underline{du}'$  given by (85) or (85'). We recapitulate this outcome by stating that the minimum-trace criterion leads to the resolution of  $a'$  and  $du'$  in the form

$$\underline{a}' = \underline{L}\underline{L}^T, \quad (90a)$$

$$\underline{du}' = \underline{L}y = \underline{a}'\underline{du}^{*'}, \quad (90b)$$

where  $y$  is given by (71) and a convenient explicit form of  $\underline{L}$  is presented in (77) together with (76).

The norm of the parametric solution  $\underline{du}'$  is defined by the square root of the product  $\underline{du}'^T \underline{du}'$ . We now address the conditions leading to a minimum norm of  $\underline{du}'$ , i.e., to  $\underline{du}'^T \underline{du}' = \text{minimum}$ . The groundwork for this task has also been laid in Section 5.3, where (86) expresses the family of  $\underline{du}'^T \underline{du}'$  in terms of the fixed part  $\underline{du}'^T \underline{du}'$  and the variable part  $(\Delta \underline{L}y)^T (\Delta \underline{L}y)$ . Since the latter is equal to the sum of squares of all the elements in the vector  $\Delta \underline{L}y$ , the necessary and sufficient condition for (86) to achieve a minimum is

$$\Delta \underline{L}y = 0. \quad (91)$$

Clearly,  $\Delta \underline{L} = 0$  alone would bring about this minimum. However, such a condition is sufficient but not necessary. It would become also necessary if (86) should be fulfilled with any possible  $y$  and thus also with any possible observational vector  $dx$ . But this is not our case, where only one vector  $dx$  is part of the adjustment model, and we search for an optimal solution within this model.

In considering  $\Delta \underline{L}$  from (68), we conclude that (91) is satisfied only if

$$\Delta \underline{L}'' y = 0. \quad (91')$$

We now partition  $\Delta \underline{L}''$  into its first column and into the remaining  $u'-1$  columns forming the submatrix  $\Delta \underline{L}''_2$ . In analogy to this, the (unique) column vector  $y$  is partitioned into its first element  $a$  and into the remaining  $u'-1$  elements  $b, \dots$  forming the vector  $y_2$ . It is assumed that  $a \neq 0$ , otherwise a different partition would take place. This is always possible unless  $y = 0$ , which would be a trivial case where  $\underline{du}^{*'} = 0$  and  $\underline{du}' = 0$  regardless of  $\underline{L}$  and  $\Delta \underline{L}$ . With the above partitions, (91') yields

$$\Delta \underline{L}'' = \Delta \underline{L}''_2 \begin{bmatrix} (1/a)y_2 & 1 \end{bmatrix}, \quad (92)$$

where the matrix  $\Delta \underline{L}''_2$  is completely arbitrary.

With (91) satisfied, it is clear from (84) that the resolution of  $\underline{du}'$  is unique, equal to  $\underline{du}'$  as in (90b). However,  $\underline{a}'$  is not unique, although it is partially restricted by (92), where only the matrix  $\Delta \underline{L}''_2$  of dimensions

$u'' \times (u' - 1)$  is arbitrary, not the matrix  $\Delta L$  of dimensions  $u'' \times u'$  as in the general case. In view of (68) and (92), the partially restricted family of  $\Delta L$  reads

$$\Delta L = [-R^T \quad I]^T \Delta L_2 [- (1/a) y_2 \quad I] . \quad (93)$$

The outcome for  $a'$  and  $du'$  is recapitulated as

$$a' = LL^T , \quad (94a)$$

$$\underline{du}' = Ly = a' du^{*'} , \quad (94b)$$

where the family of  $L$  is written as  $\underline{L} + \Delta L$  in (83), except that the family of  $\Delta L$  is now given by (93). With this restriction in mind, we can rewrite the general formulas for  $a'$  and  $\text{Tr}(a')$  from (81) and (83) as they stand.

Since the partially restricted family of  $\Delta L$  in (93) depends on the data through the column vector  $y$ , the members of the family of  $a'$  in (94a,b) could be called "data-induced inverses" of  $a^{*}$ . Even though such inverses are non-unique, they give rise to the unique canonical solution  $\underline{du}'$  whose norm is a minimum. However, unless  $\Delta L = 0$  characterizing the canonical system itself, it is apparent from (83) that the trace of the elements in each matrix of the family of  $a'$  is larger than  $\text{Tr}(\underline{a}')$ .

In principle, the minimum-norm criterion produces a unique solution and a non unique variance-covariance matrix associated with it. From the adjustment standpoint, such a situation is unacceptable. In a different approach independent of data, the minimum-trace criterion results in the unique canonical solution  $\underline{du}'$  and the unique canonical variance-covariance matrix  $\underline{a}'$  associated with it. The column vector  $\underline{du}'$  has the minimum norm and the matrix  $\underline{a}'$  has the minimum trace. These characteristics, coupled with the relative simplicity and computational efficiency of the canonical expressions, make the minimum-trace criterion a preferred tool in the resolution of rank-deficient models.

## 6. CONNECTIONS AMONG THE FORMULATIONS

### 6.1 Minimal-Constraint Matrix

Chapters 3 and 4 have resolved the geometrical setup along general lines, and have transcribed the main results in matrix notations to facilitate their comparison with standard adjustment formulas. The tensor  $\bar{Q}_{\alpha}^{\Lambda}$ , which played an important role in that development, has been transcribed as  $\bar{Q}$ , known in adjustment calculus as the minimal-constraint matrix. The geometrical development has resulted in a non-unique solution  $du'$  and a non-unique variance-covariance matrix  $a'$ . These quantities become unique if  $\bar{Q}$  is given explicitly. In practice,  $\bar{Q}$  is sought in a form that leads to desired features of the resolution. It is well known, for example, that if  $A\bar{Q}^T=0$ , the minimal-constraint formulation leads to the smallest possible trace of  $a'$ . We shall confirm this and other properties using the outcome of the preceding chapter.

Various resolution criteria, such as those involving the trace of  $a'$  or the norm of  $du'$ , are linked to the components  $\ell^{\alpha}$ ,  $j^{\alpha}$ , ... and thus to an implicit parametric-space coordinate system. These criteria cannot be represented graphically (in terms of tensor invariants), nor expressed in tensor equations, because the geometrical configuration as well as the pertinent tensor equations are independent of any coordinates. One is therefore compelled to analyze individual tensor components as we have done in Chapter 5. This step would be unnecessary in the full-rank adjustment, where  $u''=0$  and no components of the orthonormal vectors are arbitrary.

Chapter 5 has shown that  $\bar{Q}$  is not needed if the adjustment quantities  $du'$  and  $a'$  are expressed directly through the parametric-space components of  $\ell$ ,  $j$ , ..., without an intermediary such as the extension surface. The latter has entered the development, in one capacity or another, in both Chapters 3 and 4. Although this surface is implicated also in Chapter 5 through the parametric-space components of  $t$ , ..., its role could have ended upon obtaining an expression for the family of  $L$  needed to describe the families of  $du'$  and  $a'$ . Such a limited role would have merely confirmed that  $\Delta L$  in (68), based entirely on  $L^*{}^TL=I$  in (54a), is valid. However, we have developed also the family of  $T^*$ . In order to relate the analytical formulation in Chapter 5 to the formulations in Chapters 3 and 4 which utilize the matrix  $Q$ .



Table 1 suggests that a general matrix  $\bar{Q}$  can be presented as

$$\bar{Q} = \bar{T} T^* T, \quad (95)$$

where

$$\bar{T} = [[t^A] \dots], \quad (96)$$

and where  $T^*$  appears in (80). With the latter, we describe the family of  $\bar{Q}$  by

$$\bar{Q} = \bar{T} (T'')^{-1} (H^T - \Delta L'' L^* T) \quad (95')$$

Chapter 5 has shown that the only variable elements affecting  $du'$  and  $a'$  are those grouped in  $\Delta L''$ . We can thus qualify the statement at the close of Chapter 3 by stating that the variability in  $du'$  and  $a'$  caused by the variability in  $\bar{Q}$  is due to  $\Delta L''$  but not to  $T''$  or  $\bar{T}$ .

From the structure of  $\bar{Q}$  in (95') it is clear that a given minimal-constraint matrix of dimensions  $(u'' \times u)$  can be premultiplied by any regular matrix and the result is again a valid minimal-constraint matrix. Such a change in  $\bar{Q}$  can always be thought of as absorbed by the arbitrary matrices  $\bar{T}$  or  $T''$ . The latter possibility illustrates another important fact, namely that  $\bar{T}$  can vary without affecting  $\bar{Q}$  because  $T''$  can always compensate for such variations. In relation to Chapter 3, this property helps us to verify that  $k^* \equiv k^{-1}$  can indeed be an arbitrary positive-definite matrix used as a weight matrix in conjunction with a given  $\bar{Q}$ . Upon writing

$$k = \bar{T} \bar{T}^T \quad (97)$$

as suggested by Table 1, one can compute  $\bar{T}^T$ , for example, by the Choleski algorithm for positive-definite matrices. Although changes in  $k^*$  entail changes in  $\bar{T}$ , the matrix  $\bar{Q}$  can be kept intact by virtue of the arbitrariness in  $T''$ . In tracing such variations further, we realize that they propagate into  $T^*$  and thereby into  $a^*$ . (To see this we use  $k^* = k^{-1}$  in 40d and confirm that  $a^* = T^* T^{*T}$ , where  $T^*$  is affected by  $T''$  in the manner of 80.) It should be emphasized that the variations described in this paragraph leave  $du'$  and  $a'$  intact.

In assessing the approaches of Chapters 3-5, we first recall that  $du'$  and  $a'$  depend on the arbitrary matrix  $\Delta L''$  either through  $L$  or through  $\bar{Q}$ . The former case, presented in Chapter 5, is straightforward. The latter case, presented in Chapters 3 and 4, is more complex due to the introduction of  $\bar{Q}$

containing the additional arbitrary matrices  $T$  and  $T''$ . However, these two matrices have no bearing on  $du'$  and  $a'$ . From the standpoint of this study, they serve mainly to illustrate that the standard adjustment formulations using  $\bar{Q}$  are quite cumbersome when compared to the analytical formulation. If we compare the sizes of the matrices to be inverted, an economical edge of the analytical formulation becomes apparent as well. Chapter 3 contains one such matrix of dimensions  $(u \times u)$ , where  $u = u' + u''$ . Chapter 4 also contains one matrix to be inverted, but its dimensions are  $[(u + u'') \times (u + u'')]$ . However, Chapter 5 only needs to invert one matrix of dimensions  $(u' \times u')$  and one matrix of dimensions  $(u'' \times u'')$ .

Perhaps the greatest asset of the analytical formulation is the simplicity of its theory. In keeping the geometrical qualities of the basic matrices in focus, one can readily generate families of results fulfilling specific criteria and classify them according to the  $u'' \times u'$  arbitrary elements grouped in the matrix  $\Delta L''$ . Equivalent results could be obtained using  $\bar{Q}$ , but this would entail differentiation of complex matrix expressions. Such an approach, besides being more tedious, would be algebraic in nature. It has been avoided, and the outcome of Chapter 5 has been extended to benefit also the formulations in Chapters 3 and 4. This strategy is rooted in the fact that a given matrix  $\Delta L''$  leads to the same results regardless of the methodology, i.e., regardless of whether it is used in forming  $L$  in (78) or  $\bar{Q}$  in (95').

These results are now briefly summarized. A completely arbitrary  $\Delta L''$  characterizes a general resolution represented by the (unrestricted) families of  $du'$  and  $a'$ . The partially restricted  $\Delta L''$  from (92) characterizes the unique minimum norm solution  $du'$  and a partially restricted family of  $a'$ . And  $\Delta L'' = 0$  characterizes the unique minimum-norm solution  $du'$  and the unique minimum-trace variance covariance matrix  $a'$ . As is suggested by (95'), in the last case the matrix  $Q$  simplifies to the form denoted  $\bar{Q}$  and called canonical, where

$$\bar{Q} = T(T'')^{-1}H^T \quad (98)$$

The matrices  $T$  and  $T''$  used in forming  $\bar{Q}$  in all three categories are regular, but otherwise can be arbitrary. In terms of Chapter 3, this implies that the variance covariance matrix  $k - TT^T$  associated with the minimal constraints can be arbitrary provided it is positive definite.

Finally, we make a connection between the form of  $\bar{Q}$  and the rank of the matrix  $A$  augmented by  $\bar{Q}$ . In a first step, using any member of the family of  $Q$  presented in (95) as

$$\bar{Q} = \bar{T}T^{*T}, \quad (99)$$

we show that

$$\text{rank} \begin{bmatrix} A \\ \bar{Q} \end{bmatrix} = u, \quad (100)$$

which is (39b) developed in Chapter 2 through geometrical considerations. Since (99) represents a straightforward geometrical relationship, this step reduces to a confirmation of consistency in geometrical derivations. Recalling from (13a) that  $A=FL^{*T}$ , we write

$$\begin{bmatrix} A \\ \bar{Q} \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & \bar{T} \end{bmatrix} \begin{bmatrix} L^{*T} \\ T^{*T} \end{bmatrix}. \quad (101)$$

The first matrix on the right-hand side has the full column rank  $u$ , and the second matrix is regular. Upon postmultiplying (101) by  $[L \ T]$  and recalling the identity (55), the product  $[A^T \ \bar{Q}^T]^T [L \ T]$  is seen to have the full column rank  $u$ , hence (100) is necessarily true.

As an important special case of (99), we consider the canonical system synonymous with the minimum trace property. Thus,  $\bar{Q}$  is restricted to the special case seen in the explicit form in (98), written in analogy to (99) as

$$\bar{Q} = \bar{T}\underline{T}^{*T}. \quad (99')$$

(The term "case" is used here loosely, reflecting the fact that  $\Delta L''=0$ ; we know that  $\bar{T}$  and  $T''$  can be arbitrary, but these matrices hold little interest at this stage.) Upon using (13a) for  $A$  in conjunction with the identity (72b), it follows from (99') that

$$A\bar{Q}^T = 0, \quad (100')$$

which constitutes a special case of (100).

In including also a converse demonstration, we further show that (100) leads to (99). In this process the symbol  $G$  is used in place of  $\bar{Q}$ . Since it must hold that  $\text{rank}(G)=u$ , the matrix  $G$  can be decomposed into the product

$$G = G_1 G_2 .$$

where  $G_1$  is regular and  $G_2$  has the full row rank  $u''$ . The matrix  $G_1$  can be considered arbitrary, but if  $G$  is fixed then any one choice of  $G_1$  settles  $G_2$ . Recalling again (13a) for  $A$ , we can write a relation similar to (101), except that  $\bar{T}$  is replaced by  $G_1$  and  $T^{*T}$  is replaced by  $G_2$ . Since the matrix on the left-hand side of this new equation has the full column rank  $u$  by definition, the matrix  $[L^* G_2^T]^T$  on the right-hand side must be regular. But then  $G_2$  can represent an admissible matrix  $T^{*T}$  (which, in turn, settles  $L$  and  $T$ ). Since we can further identify  $G_1$  with the arbitrary regular matrix  $\bar{T}$ , we can write  $G = \bar{T} T^{*T}$  for any matrix  $G$  satisfying the initial rank stipulation. Accordingly, any admissible matrix  $G$  has the structure of  $\bar{Q}$  in (99), and the demonstration is terminated.

As a special case in the converse demonstration, we wish to proceed from (100') to (99') using the same notations and basic steps. We thus begin with  $AG^T = 0$ , representing a special case of the initial rank stipulation. Next, we replace  $A$  by  $FL^{*T}$  according to (13a). Since the rank of  $F$  is  $u'$  it follows that  $L^{*T}G^T = 0$ , hence  $L^{*T}G_2^T = 0$ , which is similarly a special case showing that the matrix  $[L^* G_2^T]^T$  is regular. But from (72b) it follows that the admissible  $T^{*T}$  represented by  $G_2$  must be  $\underline{T}^{*T}$ , so that we can write  $G = \bar{T} \underline{T}^{*T}$  completing the demonstration. We have thus established that in the approaches utilizing the matrix  $\bar{Q}$ , the minimum trace criterion is satisfied only if  $A\bar{Q}^T = 0$ . This occurs only for  $\bar{Q} = \underline{Q}$ , whose explicit form is given by (98). In practice, such a matrix is usually easy to obtain from the structure of the adjustment model.

## 6.2 Confirmation of Consistency

In this section, we shall verify that the results for  $du'$  and  $a'$  are the same in all three formulations described by Chapters 3-5. For its simplicity, the standard of comparison is adopted from Section 5.1 in the form of equations (60), (59), and (61), respectively:

$$du' = Ly, \quad a' = LL^T, \quad (102a,b)$$

where the fixed column vector is

$$y = L^T du^{*'} . \quad (102c)$$

Equations (102a,b) are general, not subject to any criteria such as those analyzed in the latter part of Chapter 5. They are also the most straightforward expressions offered by the geometrical setup for  $du'$  and  $a'$ .

The general results for  $du'$  and  $a'$  derived in Chapter 3 appear in (41a,b), rewritten here as

$$du' = a du^{*'} , \quad a' = aa^{*'}a . \quad (103a,b)$$

The matrix  $a$  is given by (42), which reads

$$a = (a^*)^{-1} . \quad (103c)$$

The matrix  $a^*$  is formed as in (40b-d), namely

$$a^* = a^{*'} + a^{*''} , \quad a^{*'} = A^T g^* A , \quad a^{*''} = \bar{Q}^T k^* \bar{Q} . \quad (103d,e,f)$$

All of (103a-f) follow from the geometrical setup developed in the context of the universal space.

Finally, the general results for  $du'$  and  $a'$  derived in Chapter 4 can be read from (49), already confirming (102a,b). However, this confirmation is achieved through the matrix  $\bar{Q}$ , which we shall not consider as a whole, but, rather, in terms of the basic matrices  $T$  and  $T^*$ . (The same applies for  $\bar{Q}$  in the last paragraph.) Thus, we begin with equation (48), rewritten in the form

$$\begin{bmatrix} du' \\ dw^* \end{bmatrix} = \begin{bmatrix} a^{*'} & \bar{Q}^T \\ \bar{Q} & 0 \end{bmatrix}^{-1} \begin{bmatrix} du^{*'} \\ 0 \end{bmatrix} . \quad (104)$$

After the indicated inversion, the matrix  $a'$  will occupy the location of  $a^{*'}.$  Equation (104) is a consequence of the geometrical setup developed in terms of an extension-surface vector restricted to zero.

Since all of (102a)-(104) reflect the same geometrical setup, the results for  $du'$  and  $a'$  must be identical. Although this is not readily apparent, we should be able to confirm it using known geometrical relations among the orthonormal vectors  $i, j, \dots, t, \dots$ . The parametric-space components of these vectors have been grouped in the matrices  $L^*, L, T^*,$  and  $T$  forming a general system. Accordingly, the known geometrical relations just mentioned are expressed by (53), or (54a-d), or (55), which are all equivalent. However, the presence of  $\bar{Q}$  requires the use of two additional matrices, denoted  $\bar{T}$  and  $\bar{T}^*$ , which contain the extension-surface components of  $t, \dots$ . The former has been presented in (96), while the latter is similarly defined as

$$\bar{T}^* = [[t_\Lambda] \dots] \quad (105)$$

We now list the geometrical identities which will be used in relating the results of Chapters 3 and 4 to those of Chapter 5. From Table 1 we read

$$a^{*'} = L^* L^{*T}, \quad a^{*''} = T^* T^{*T}; \quad (106a,b)$$

$$\bar{Q} = \bar{T} T^{*T} \quad (107)$$

Equations (106a,b) also follow from (103e,f) upon using the decomposition of  $A$  and  $\bar{Q}$  (the latter equation further uses  $k^* = k^{-1}$  with  $k$  given in 97). Although all three identities have already been encountered, they are listed here for an easy reference. The definitions (96) and (105) imply the identity  $T^* \bar{T}^T = I$ , and thus

$$\bar{T}^{*T} = \bar{T}^{-1} \quad (108)$$

Finally, due to the orthogonality of any vectors in the extension surface with respect to any vectors in the model surface, we have as a special case:

$$T^T du^{*'} = 0 \quad (109)$$

This identity also follows from (40e), (13a), and (54b).

In turning to the outcome of Chapter 3 as presented in (103a-f), we write

$$a^* = [L^* \ T^*][L^* \ T^*]^T, \quad a = [L \ T][L \ T]^T, \quad (110a,b)$$

where (110a) follows from (103d) and (106a,b), while (110b) follows from (103c) and (55). Both identities can also be deduced from Table 1. If we substitute (110b) in (103a) and use (102c) and (109), we confirm the result (102a). And if we substitute (106a) with (110b) in (103b), we confirm (102b) as well.

The confirmation of the outcome from Chapter 4 is lengthier and more complex. First, we form the matrix to be inverted in (104) as

$$\begin{bmatrix} a^* & \bar{Q}^T \\ Q & 0 \end{bmatrix} = \begin{bmatrix} L^* & T^* & 0 \\ 0 & \bar{T} & \bar{T} \end{bmatrix} \begin{bmatrix} L^* & 0 & T^* \\ 0 & \bar{T} & -\bar{T} \end{bmatrix}^T, \quad (111)$$

which can be verified upon using (106a) and (107). Next, we invert the matrices on the right hand side as follows:

$$\left\{ \begin{bmatrix} L^* & 0 & T^* \\ 0 & \bar{T} & -\bar{T} \end{bmatrix}^T \right\}^{-1} = \begin{bmatrix} L & T & T \\ 0 & \bar{T}^* & 0 \end{bmatrix}, \quad (112a)$$

$$\begin{bmatrix} L^* & T^* & 0 \\ 0 & \bar{T} & \bar{T} \end{bmatrix}^{-1} = \begin{bmatrix} L & T & -T \\ 0 & 0 & \bar{T}^* \end{bmatrix}^T, \quad (112b)$$

where (112a) can be verified via (54a-d), and (112b) can be verified via (53).

If we multiply together the right hand sides of (112a,b) in the same order, we obtain the inverse of the matrix on the left hand side of (111). The substitution of the latter in (104) yields

$$\begin{bmatrix} du \\ dw^* \end{bmatrix} = \begin{bmatrix} LL^T & T\bar{T}^{*T} \\ T^*T^T & 0 \end{bmatrix} \begin{bmatrix} du^* \\ 0 \end{bmatrix}. \quad (113)$$

In considering (102c), the first equation implied by (113) confirms (102a). Moreover, with (109) taken into account, the second equation yields  $dw^* = 0$ . In agreement with the statement following (104), the location of  $a^*$  in (113) is occupied by  $LL^T$ , which confirms also (102b). Upon further scrutiny, (113) is seen to be identical to (49) due to the identity

$$A = T\bar{T}^{*T}, \quad (114)$$

which can be read directly from Table 1. This concludes the task of verifying the consistency among the results of Chapters 3-5.

## 7. DISCUSSION

### 7.1 Completeness of Adjustment Families

The method suggested in Section 5.1 for the computation of  $L^{*'} from  $N_{11}$  has been the Choleski algorithm for positive-definite matrices. Subsequently, the matrix  $L^*$  in (57) has been assumed fixed during the development leading to the description of adjustment families in Section 5.3. We shall now show that any method used in the computation of  $L^{*}$  results in the same adjustment families of  $du'$  and  $a'$ , which are thus complete regardless of the numerical values in  $L^{*}$ . The only admissibility condition for  $L^{*}$  is$

$$L^{*'} L^{*'}{}^T = N_{11} , \quad (115)$$

which has also been at the root of the Choleski choice in Section 5.1. Since  $N_{11}$  is given and thus fixed, the matrix  $R$  defined in (58) is fixed throughout, i.e., independent of  $L^{*}$ . The geometrical relations leading to (61) have revealed that the column vector  $y$  is fixed as well. Furthermore, in consulting (68), we realize that although  $\Delta L$  is variable as a function of  $\Delta L''$ , it is fixed with respect to  $L^{*}$ .

On the other hand,  $\underline{L}$  and thus also  $L$  depend on  $L^{*}$ , as is clear from (74) or (77) and from (78). However, if we form  $\underline{a}'$  by (82) in conjunction with (74), we obtain

$$\underline{a}' = \underline{L} \underline{L}^T = [I \ R]^T (I + R R^T)^{-1} N_{11}^{-1} (I + R R^T)^{-1} [I \ R] ,$$

where (115) has been taken into account. Moreover, in rewriting (85') as

$$\underline{du}' = \underline{a}' du^{*'} ,$$

where  $du^{*}$  is given and thus fixed, we conclude that both  $\underline{du}'$  and  $\underline{a}'$  are fixed. Accordingly, (83) and (86) indicate that the trace of  $a'$  and the norm of  $du'$  are independent of  $L^{*}$ , although the same cannot be said of the families of  $a'$  and  $du'$  themselves. But if one can show that the family of  $a'$  is independent of  $L^{*}$ , the same will be true for the family of  $du'$  by virtue of (60').

The family of  $a'$  is given by (81), where the first two terms on the right hand side are independent of  $L^{*}$ . Since the third and the fourth terms are transposes of each other, all we need to show is that  $\underline{L} \Delta L^T$  is independent of  $L^{*}$ . We begin by assuming the existence of two distinct families of  $L$ :  $\underline{L}$  and  $\underline{L} + \Delta \underline{L}$ .



The first corresponds to the Choleski choice and is called "i", and the second corresponds to a different admissible choice and is called "j". Based on (115), it must hold that

$$L_i^{*'} L_i^{*'}^T = L_j^{*'} L_j^{*'}^T, \quad (116a)$$

where  $L_i^{*'}$  is the submatrix  $L^{*'}$  as evaluated in the family i, etc. Next we postulate the following admissible relationship between the unrestricted submatrices  $\Delta L_i''$  and  $\Delta L_j''$ :

$$\Delta L_i'' = \Delta L_j'' (L_j^{*'})^{-1} L_i^{*'} \quad (116b)$$

Since both  $L_i^{*'}$  and  $L_j^{*'}$  are regular, any  $\Delta L_j''$  can be written as

$$\Delta L_j'' = \Delta L_i'' (L_i^{*'})^{-1} L_j^{*'} \quad (116c)$$

where the first submatrix exists (in fact, it can be computed by 116b).

As a matter of interest, we form  $\Delta L \Delta L^T$  for the family j, where  $\Delta L$  is given by (68) with  $\Delta L''$  attributed the subscript j. In taking advantage of (116b) and then of (116a), we arrive at  $\Delta L \Delta L^T$  for the family i. This outcome is obtained with any possible  $\Delta L_j''$  in conjunction with a given  $L_j^{*'}$ , and, upon repeating the procedure, in conjunction with any admissible  $L_j^{*'}$ . This indicates that the family i for  $\Delta L \Delta L^T$  is indeed complete. Since the derivation could be retraced with "i" and "j" interchanged, it follows that any admissible  $L^{*'}$  leads to the same (complete) family of  $\Delta L \Delta L^T$ . Essentially the same conclusion has been reached earlier, when  $\Delta L$  has been shown to be independent of  $L^{*'}$ .

The crucial step in the current demonstration proceeds in analogy to the previous paragraph, but is simpler in that it does not necessitate the identity (116a). We form  $\underline{L} \Delta L^T$  for the family j, where both the submatrix  $L^{*'}$  in (74) or (77) and the submatrix  $\Delta L''$  in (68) are attributed the subscript j. Keeping  $\underline{L}$  intact while taking advantage of (116b), we arrive at  $\underline{L} \Delta L^T$  for the family i. An argument similar to the one above leads to the conclusion that the family i for  $\underline{L} \Delta L^T$  is complete, and that the same family can be obtained with any admissible  $L^{*'}$ . This concludes the proof that the families of  $du'$  and  $a'$  are independent of  $L^{*'}$ , i.e., that the same (complete) families are obtained with any  $L^{*'}$  fulfilling (115). One can thus adopt the Choleski choice and consider  $L^{*'}$  and thereby  $L^*$  fixed throughout, as has been anticipated in Section 5.1.

## 7.2 Uniqueness of the Canonical Property

The property central to the analysis of adjustment quantities in Chapter 5 has been embodied by (73a) holding identically with any admissible  $\Delta L$ , namely

$$\underline{L}^T \Delta L = 0 \quad (117)$$

As we have seen in Section 5.2, any matrix  $L$  can be expressed by  $L = \underline{L} + \Delta L$ , where the first member is defined in (65) and the second member appears in (68).

Suppose now that  $\underline{L}$  is not the only matrix fulfilling (117) and denote another such matrix as  $\underline{L}_1$ , where

$$\underline{L}_1 = \underline{L} + \Delta \underline{L} \quad (118)$$

Since  $\underline{L}_1$  is a matrix from the family of  $L$ ,  $\Delta \underline{L}$  must be expressible by (68), i.e., it must have the form

$$\Delta \underline{L} = [R^T \quad I]^T \Delta L'' \quad .$$

If it should hold that  $\underline{L}_1^T \Delta L = 0$  for any admissible  $\Delta L$ , in considering (118) together with (117) we would have

$$\Delta \underline{L}_1^T \Delta L = 0 \quad ,$$

where  $\Delta L$  has the form (68). This relation can thus be written explicitly as

$$\Delta \underline{L}_1^T (I + R^T R) \Delta L'' = 0 \quad , \quad (119)$$

where the submatrix  $\Delta L''$  of dimensions ( $u'' \times u'$ ) is completely arbitrary, and the submatrix  $\Delta \underline{L}_1''$  of the same dimensions is to be determined. In the usual situation, where  $u' > u''$ ,  $\Delta L''$  can be thought of as partitioned into a ( $u'' \times u''$ ) submatrix and the remaining submatrix. Since the former can be assumed nonsingular, and since the matrix  $(I + R^T R)$  is positive definite, it follows from (119) that we must have  $\Delta \underline{L}_1'' = 0$ , hence

$$\Delta \underline{L}_1 = 0 \quad (120)$$

For the sake of completeness, we also consider the unlikely case  $u' < u''$ . Since (119) must be satisfied with any  $\Delta L''$ , the net effect is the same as if it had to hold with all such submatrices simultaneously, i.e., as if the columns of  $\Delta L''$  were augmented by the columns of all the other submatrices  $\Delta L''$ . One can then always choose  $u''$  independent columns from their totality, and form a

regular matrix. In so doing, one arrives at the last step in the preceding paragraph, including the result (120). We can thus conclude that the matrix  $\underline{L}$  fulfilling (117) is unique, defined by (65) and developed into (74) and further into (77). Indeed, this is the matrix we would obtain if we started from the condition (117).

### 7.3 Canonical System and the Pseudoinverse

The unique pseudoinverse  $M^+$  of the matrix  $M$  fulfills four conditions, called "l", "g", "r", and "m". The condition "l" states that the matrix  $MM^+$  is symmetric, "g" states that  $MM^+M=M$ , "r" states that  $M^+MM^+=M^+$ , and "m" states that  $M^+M$  is symmetric. In the present context, the role of  $M$  is played by  $L^*{}^T$ . In considering  $L^*{}^TL=I$  from (54a), we observe that the condition "l" for  $L$  as a potential pseudoinverse of  $L^*{}^T$  is satisfied. The consequence of this special form of "l" is an automatic fulfillment of the conditions "g" and "r" as well. On the other hand,  $LL^*{}^T$  is symmetric only if  $L=\underline{L}$ . In particular, with  $\underline{L}$  from (65) the symmetry is confirmed and thus

$$\underline{L} = L^*(L^*{}^TL^*)^{-1} = (L^*{}^T)^+ \quad (121)$$

There can be no other matrix  $L$  satisfying the condition "m" because the pseudoinverse is unique.

In considering the identities (56) and (59), i.e.,

$$a^{*'} = L^*L^*{}^T, \quad a' = LL^*{}^T,$$

with the aid of (54a) we deduce that

$$a^{*'}a'a^{*'} = a^{*'}, \quad a'a^{*'}a' = a'.$$

These equations follow also from the geometrical setup, and can be obtained upon consulting Table 1. If  $a'$  should be the pseudoinverse of  $a^{*'}$ , the matrix  $LL^*{}^T$  would have to be symmetric, in which case both conditions "l" and "m" would be satisfied simultaneously. But this can occur only if  $L=\underline{L}$ , as we gather from the last step that has led to (121). Accordingly, we conclude that

$$\underline{a}' = \underline{LL}^*{}^T = (a^{*'})^+ \quad (122)$$

As a matter of interest, we now turn our attention to the matrices  $T^*$ ,  $T$ ,  $\bar{Q}$ , and  $\bar{A}$ . In repeating the demonstration which has resulted in (121), but with  $T^*$  replacing  $L^*$  and (51d) replacing (54a), we observe that  $T$  and  $T^{*T}$  are the pseudoinverses of each other only if  $TT^{*T}$  is symmetric, which, in turn, occurs only if  $T^* = \underline{T}^*$ . Paralleling (121), we thus have

$$T = \underline{T}^* (\underline{T}^{*T} \underline{T}^*)^{-1} = (\underline{T}^{*T})^+ , \quad (123)$$

where the first equality follows from (66). The matrix  $\bar{Q}$  from (95) and the matrix  $\bar{A}$  from (114) yield  $\bar{Q}\bar{A} = I$ , hereby satisfying the first three criteria for  $\bar{A}$  to be the pseudoinverse of  $\bar{Q}$ . However, the condition "m" can be satisfied only if  $TT^{*T}$  is symmetric, which leads to  $T^* = \underline{T}^*$  as above and thus to the canonical system. The last assertion is due to the fact that  $\Delta T^* = 0$  in (69) entails  $\Delta L^* = 0$ . (The canonical property does not depend on  $T^*$ , but the latter must be the same in both  $\underline{T}^*$  and  $T$ .) In this system one has

$$\bar{Q} = \underline{T} \underline{T}^{*T} , \quad \bar{A} = \underline{T} \underline{T}^{*T} , \quad (124a,b)$$

where  $\underline{T}^*$  and  $T$  are linked by (123). With the aid of (124a,b) and (108), we finally write

$$\bar{A} = \bar{Q}^T (\bar{Q} \bar{Q}^T)^{-1} = \bar{Q}^+ . \quad (125)$$

In terms of Chapter 4, we conclude that if the canonical system is implied in (48) and (49), not only is the minimum-trace criterion satisfied, but the four corresponding matrices in these equations are the pseudoinverses of each other.

## 8. SUMMARY AND CONCLUSIONS

The subject of this study is the parametric least-squares (L.S.) method, where the adjustment model is either linear or has been linearized beforehand. In adjustment notations, the L.S. setup is represented by

$$L = AX + V,$$

where  $L$ ,  $V$ , and  $X$  are the vectors of  $n$  linearized observations,  $n$  residuals, and  $u$  parametric corrections to  $X^0$  (an initial set of parameters), respectively. The vector  $L$  is formed as  $L^b - L^0$ , where  $L^b$  contains the actual observations and  $L^0$  contains the values of the observables consistent with  $X^0$ . An additional vector representing  $n$  adjusted linearized observations is symbolized by  $L'$ , where

$$L' = AX.$$

The letter  $A$  denotes the design matrix of dimensions  $(n \times u)$ , which either has the full column rank or is rank-deficient. The former kind has been treated in [Blaha, 1984], while the latter is dealt with herein. However, the basic L.S. setup and its geometrical interpretation are the same in both the full-rank and the rank-deficient adjustment models. The L.S. criterion in either model reads

$$V^T P V = \text{minimum}, \quad P = \Sigma^{-1},$$

where  $P$  and  $\Sigma$  are respectively the weight matrix and the variance covariance matrix of observations, both of which are positive-definite. The quantities  $\Sigma$  (and thus  $P$ ),  $A$ , and  $L$  are known a priori.

Most of the theory concerned with the development of a geometrical setup isomorphic in every respect to the parametric L.S. adjustment can be found in Chapter 2. In this task, the tensor structure has proven invaluable. It has brought about simple correspondences between adjustment quantities and geometrical objects, as can be gathered upon transcribing the above three equations in tensor notations:

$$dx^r = A_{\alpha}^r du^{\alpha} + dx^{r0},$$

$$dx^{r0} = A_{\alpha}^r du^{\alpha},$$

$$dx^S g_{Sr} dx^{r0} = \text{minimum}, \quad g_{st} g^{rt} = \delta_s^t.$$

Accordingly,  $dx^r$  corresponds to the (linearized) observations,  $dx'^r$  to the adjusted (linearized) observations,  $dx''^r$  to the error estimates (i.e., minus the residuals),  $du'^\alpha$  to the parametric corrections,  $A_\alpha^r$  to the design matrix,  $g_{sr}$  to the weight matrix of observations, and  $g^{rt}$  to the variance-covariance matrix of observations. Just as their adjustment counterparts, the tensors  $g^{rt}$  (and thus  $g_{sr}$ ),  $A_\alpha^r$ , and  $dx^r$  are known a priori.

The geometrical objects whose components are  $dx^r$ ,  $dx'^r$ ,  $dx''^r$ , and  $du'^\alpha$  are referred to as vectors  $dx$ ,  $dx'$ ,  $dx''$ , and  $du'$ , respectively. Although  $A_\alpha^r$  cannot itself be represented graphically by a single object, it can be expressed via components of orthonormal vectors. In its tensorial formulation, the L.S. criterion stipulates that the length of the vector  $dx''$  must be a minimum. Since the vector  $dx'$  is restricted to an implied model surface, it follows that  $dx''$  is orthogonal to this surface. Crucial to the geometrical development of both the full-rank and the rank-deficient adjustment models is the property whereby  $A_\alpha^r$  transforms the components of vectors lying in the model surface from one coordinate system to another. Accordingly, the above relation for  $dx'^r$  indicates that the vector denoted  $du'$  is identical to  $dx'$ .

The mere transcription of the basic L.S. setup in tensor notations suggests an alternative to the standard algebraic treatment of the parametric adjustment. The approach undertaken herein uses geometry with tensor structure to express all the adjustment quantities in terms of orthonormal vectors. Through this isomorphism, the observational vector  $dx$  lying in an  $n$ -dimensional observational space is projected onto a  $u'$ -dimensional model surface as  $dx' = du'$ . The observational space is spanned by  $n$  orthonormal vectors  $i, j, \dots, \nu, \dots$ , while the model surface embedded in this space is spanned by  $u'$  orthonormal vectors  $i, j, \dots$ . In the rank-deficient problems, where the rank of the design matrix is  $u'$  and the rank deficit is  $u'' = u - u'$ , the model surface is also embedded in a  $u$  dimensional parametric space spanned by  $u$  orthonormal vectors  $i, j, \dots, t, \dots$  and is thus an intersection of the observational and the parametric spaces. In the full-rank problems, where  $u = u'$ , the parametric space and the model surface coincide. In both cases the contravariant components  $dx'^r$ ,  $r = 1, 2, \dots, n$ , and the contravariant components  $du'^\alpha$ ,  $\alpha = 1, 2, \dots, u$ , are related through the design tensor  $A_\alpha^r$ . The latter is shown to be expressible by the observational space contravariant components and the parametric space covariant components of the orthonormal vectors  $i, j, \dots$  spanning the model surface.

The isomorphic geometrical setup reveals that all the adjustment matrices, i.e., the design matrix, the variance covariance matrices (of observations, adjusted observations, residuals, and parameters), and the corresponding weight matrices, can be expressed as a product of two constituent matrices each. This outcome is further qualified as follows:

- (a) All constituent matrices are written in terms of orthonormal vectors, the elementary geometrical objects;
- (b) These vectors are the same in either matrix of the constituent pair, only the type of their components may differ; and
- (c) The set  $i, j, \dots$  spanning the model surface is common to all constituent matrices except those pertaining to the residuals (and, in the rank-deficient context, also to the minimal constraints if the latter are used).

The geometrical development yields simple expressions for the (singular) weight matrices of adjusted observations and residuals, not derived in standard adjustment literature. In the case of rank-deficient adjustment, it confirms the familiar outcome that the unique variance-covariance matrix of parameters which has the minimum trace is the pseudoinverse of the (singular) matrix of normal equations. At the same time, the minimum-trace criterion results in the unique parametric solution which has the minimum norm. By comparison, the minimum norm criterion alone leads to a new family of "data-induced" inverses, each of which produces the same unique solution as above, not merely the same minimum norm. However, since each such inverse constitutes a variance-covariance matrix of parameters, and since a unique parametric solution with a non unique variance covariance matrix has little practical value, the minimum-trace approach is preferred to the minimum norm approach.

From the theoretical standpoint, both the minimum-trace and the minimum-norm resolutions are merely special cases of the general resolution, where the solution vector as well as its variance covariance matrix are non unique. The general resolution has been analyzed in three distinct formulations presented in Chapters 3-5, all of which have been confirmed in Chapter 6 to give identical results. This outcome is summarized below using the familiar matrix symbolism. However, the transcription of tensor relations into matrix relations is intended to preserve the letter symbols given to tensor quantities, rather than to change them to standard adjustment symbols such as those seen in the beginning

paragraph. The tensor indices are simply dropped, and, in the case of a purely covariant tensor, the original letter symbol is attributed "\*". Thus, for example, the tensor components  $dx^T$  are grouped in the adjustment (column) vector  $dx$ . Although the symbol  $dx$  has been used earlier to identify a geometrical object, its role is clearly discernible from the context. As another example, the tensor components  $du'^\alpha$  are grouped in the adjustment vector  $du'$ , while the components  $du'_\beta$ , where  $du'_\beta = A^s_{\beta sr} dx^r$ , are grouped in the adjustment vector  $du^{*'}$ , where  $du^{*'} = A^T g^* dx$ . The latter is sometimes referred to as the right hand side of normal equations. This transcription illustrates the close correspondence between tensor contractions and matrix multiplications.

The universal space formulation of Chapter 3 generates the augmented observation equations presented in (38) by joining the minimal constraint matrix  $\bar{Q}$  to the design matrix  $A$ :

$$\begin{bmatrix} dx \\ 0 \end{bmatrix} = \begin{bmatrix} A \\ \bar{Q} \end{bmatrix} du' + \begin{bmatrix} dx'' \\ 0 \end{bmatrix}.$$

The augmented matrix  $[A^T \bar{Q}^T]^T$  must have the full column rank as stipulated by (39b). The complete weight matrix is written as  $\text{diag.}(g^*, k^*)$ , where  $g^*$  is the original weight matrix of observations and  $k^*$  is an arbitrary positive definite weight matrix associated with the minimal constraints. The matrix of normal equations in this formulation is  $a^*$ , which is positive-definite in contrast to the original positive semi definite matrix  $a^{*'}$ . The general solution  $du'$  and its variance covariance matrix  $a'$  are expressed from (41a,b) as

$$du' = a du^{*'} = a' du^{*'},$$

$$a' = a a^{*'} a,$$

where  $a = (a^*)^{-1}$ . The solution  $du' = a du^{*'}$  is the standard outcome of a full rank model represented here by the augmented observation equations. However, the variance covariance matrix in such a model would be  $a$ , which should be modified as indicated in order to yield the desired positive semi definite matrix  $a'$ .

The minimal constraint formulation of Chapter 4 also uses the matrix  $Q$ , but proceeds through augmented normal equations as in the standard adjustment with (absolute) constraints. These normal equations are depicted in (48), and are resolved in the form



$$\begin{bmatrix} du' \\ dw^* \end{bmatrix} = \begin{bmatrix} a^{*'} & \bar{Q}^T \\ \bar{Q} & 0 \end{bmatrix}^{-1} \begin{bmatrix} du^{*'} \\ 0 \end{bmatrix}.$$

After the inversion, the variance-covariance matrix  $a'$  occupies the location of  $a^{*'}$ . As has been shown in (50a,b), the complete solution consists of  $du' = a' du^{*'}$  confirming the previous result, and of  $dw^* = 0$ . The vector  $dw^*$  corresponds to the Lagrange multipliers in adjustment calculus. The geometrical interpretation in Chapter 4 illustrates why they must be zero.

The analytical formulation of Chapter 5 circumvents the use of minimal constraints in any capacity, and thus also the use of  $\bar{Q}$ . Instead, it proceeds to form the basic matrices  $L^*$ ,  $L$ ,  $T^*$ , and  $T$ , containing the parametric-space components of the orthonormal vectors  $\ell$ ,  $j$ , ...,  $t$ , ... The geometrical interpretation of these matrices reveals that only  $L^*$  and  $L$ , containing the components of  $\ell$ ,  $j$ , ..., participate in the resolution of  $du'$  and  $a'$ . On the other hand,  $T^*$  and  $T$ , containing the components of  $t$ , ..., serve in the analysis of  $\bar{Q}$  in view of Chapters 3 and 4. The simplest expressions for  $du'$  and  $a'$  follow from the geometrical setup as

$$du' = a' du^{*'},$$

$$a' = LL^T.$$

A complete description of  $L$  is offered by (78), namely

$$L = \{ [I \ 0]^T + HR^T \} (L^{*'}^T)^{-1} + [-R^T \ I]^T \Delta L''.$$

With the exception of  $\Delta L''$ , all the matrices on the right-hand side can be computed from the matrix of normal equations  $a^{*'}$ , and are considered fixed. In this task,  $a^{*'}$  is partitioned clockwise into  $N_{11}$ ,  $N_{12}$ ,  $N_{22}$ , and  $N_{12}^T$ , where the leading submatrix  $N_{11}$  of dimensions  $(u' \times u')$  can always be assumed positive definite, whether a priori or upon reordering the parameters. The matrices  $(L^{*'}^T)^{-1}$  and  $R$  of dimensions  $(u' \times u')$  and  $(u' \times u'')$ , respectively, can then be obtained via the Choleski algorithm as explained in Section 5.1, i.e., via  $L^{*'} L^{*'}^T N_{11}^{-1}$  and  $R = N_{11}^{-1} N_{12}$ . Finally, the matrix  $H$  of dimensions  $(u \times u'')$  is constructed in (76) as  $H = [-R^T \ I]^T (I + R^T R)^{-1}$ .

The above expressions for  $du'$ ,  $a'$ , and  $L$  reveal that the properties of the resolution depend entirely on the  $u' \times u'$  elements grouped in  $\Delta L''$ . If this matrix is completely arbitrary,  $L$  has the most general form resulting in the

general resolution of  $du'$  and  $a'$ . Clearly, a non-unique solution  $du'$  and a non-unique variance covariance matrix  $a'$  hold mainly theoretical interest. If  $\Delta L''$  is partially restricted by (91'), the solution  $du'$  is unique and has the minimum norm. The restriction (91') can be written explicitly as

$$\Delta L'' (L^*)^{-1} \{ [I \ 0] + R H^T \} du^* = 0 ,$$

or can be further developed into (92), etc., all of which are data dependent due to  $du^*$ . However,  $a'$  is still non unique since it is only partially restricted with respect to its general counterpart. Finally, if  $\Delta L'' = 0$ , both  $du'$  and  $a'$  are unique. In this case  $du'$  is the minimum norm solution identical to the one above, and  $a'$  is its variance covariance matrix of the smallest possible trace. The three cases just described have been derived in conjunction with no criterion, with the minimum norm criterion, and with the minimum trace criterion, respectively. As has been suggested earlier, the minimum trace criterion is superior to the minimum norm criterion, and, by the same token, to any other criteria. Even if some of them produced unique  $du'$  and  $a'$ , the norm of  $du'$  would not be a minimum, or the trace of  $a'$  would not be a minimum, or both. The analytical formulation has revealed another advantage of the minimum-trace criterion over any other, namely that this criterion results in the simplest expressions possible, represented by  $\Delta L'' = 0$ .

The analysis of Chapter 5 can be extended to the approaches of Chapters 3 and 4, developed with the aid of the matrix  $\bar{Q}$ . This goal is addressed in Chapter 6, where (95') presents  $Q$  in the general form

$$Q = T(T'')^{-1} (H^T - \Delta L'' L^{*T}) .$$

The matrices  $H$  and  $\Delta L''$  have been described above, the matrix  $L^*$  of dimensions  $(u \times u')$  is given by (57) as  $L^* = [I \ R]^T L'^*$ , and the matrices  $T$  and  $T''$  of dimensions  $(u'' \times u'')$  are both arbitrary, subject only to the restriction that they must be regular. The geometrical construction of  $Q$  ensures that the rank condition (39b) is satisfied. The properties of the resolution are determined by  $\Delta L''$  in the manner of the preceding paragraph. Thus, for example, the minimum trace criterion corresponds to  $\Delta L'' = 0$ . This stipulation is equivalent to  $AQ^T = 0$ , representing a special case of the rank condition (39b). Again, the matrix  $Q$  with  $\Delta L'' = 0$  is the simplest of its kind. It is usually easy to obtain in practice from the structure of the adjustment model, without the need for an explicit formula.

The foregoing has illustrated the theoretical benefits offered by the analytical formulation of Chapter 5. However, this formulation holds also an economical edge over the other two. In particular, the analytical approach calls for an inversion of one matrix of dimensions  $(u' \times u')$  in the computation of  $(L^{*'}L^{*'})^{-1}$  and  $R$ , and one matrix of dimensions  $(u'' \times u'')$  in the computation of  $H$ . Since both matrices to be inverted are positive definite, the Choleski algorithm is well suited for this task. Although the matrix  $L^{*'}L^{*'}^T$  in itself could be regarded as non unique, Section 7.1 shows that such a variability is inconsequential because the adjustment results for  $du'$  and  $a'$  contain only the inverse of the product  $L^{*'}L^{*'}^T$ , which is unique and known. By comparison, the use of augmented observation equations in Chapter 3 requires an inversion of the positive definite matrix  $a^*$  of dimensions  $(u \times u)$ , where  $u = u' + u''$ . Finally, the use of augmented normal equations in Chapter 4 requires an inversion of a matrix whose dimensions are  $[(u + u'') \times (u + u'')]$ . Another drawback of this standard procedure is that the matrix to be inverted is not positive definite.

Although  $du'$  and  $a'$  are in general non-unique, the vector  $dx'$  containing the adjusted (linearized) observations and the variance-covariance matrix  $g'$  of these quantities are unique, expressed by

$$\begin{aligned} dx' &= Adu' \\ g' &= Aa'A^T \end{aligned}$$

These results, derived by geometrical means in Chapter 2, are applicable to each of the three formulations of Chapters 3-5. In terms of Chapter 3, the same positive semi definite matrix  $g'$  is obtained also as  $g' = AaA^T$ . The positive semi definite weight matrix associated with  $dx'$  follows from the weight propagation law established in Chapter 2 as

$$g^{*'} = g^*g'g^*$$

With regard to adjusted (linearized) functions of parameters, the pattern presented above in conjunction with  $dx'$  and  $g'$  applies in every respect, except that  $dx'$ ,  $g'$ , and  $A$  are attributed the symbol  $\hat{\phantom{x}}$ . This outcome represents, in fact, the variance covariance propagation law known from adjustment calculus. (The weight matrix  $\hat{g}^{*'}$  for such functions could not be computed in analogy to  $g^{*'}$  because  $\hat{g}^*$  is unknown.) The new weight propagation law applied to adjusted

observations could be useful if the latter, or their subset, should participate in some capacity in another adjustment. The natural and nearly effortless derivation of this law highlights potential benefits of an isomorphic approach, even though the disciplines being related may initially seem quite disparate.

Several other topics related to geometry with tensor structure have also been addressed in this study. For example, Section 7.3 confirms that the matrix  $a'$  in the minimum trace approach is the pseudoinverse of  $a^{*'}.$  Upon substituting  $AL''=0$  in  $L$  listed earlier, this pseudoinverse is expressed by

$$(a^{*'})^+ = \{[I \ 0]^T + HR^T\}N_{11}^{-1}\{[I \ 0] + RH^T\}.$$

Such an algorithm could be useful not only for its clearcut geometrical interpretation, but also for its computational efficiency. As has been already indicated, the pseudoinverse of a positive semi-definite matrix of dimensions  $[(u'+u'') \times (u'+u'')]$  entails here only one inversion of a positive definite matrix of dimensions  $(u' \times u')$  and one inversion of a positive definite matrix of dimensions  $(u'' \times u'').$  Although two positive definite matrices of dimensions  $(u' \times u')$  could be inverted instead, the above procedure is more advantageous when  $u' > u'',$  which is by far the most prevalent situation in actual rank-deficient adjustments.

A part of the theory developed in this study is illustrated with the aid of a simple example in Appendix 1. This example treats the general adjustment resolution, the minimum norm resolution, and the minimum trace resolution in each of the three formulations presented in Chapters 3-5. Appendix 2 introduces a commutative diagram corresponding to Table 1 in Chapter 2, which offers a visual representation of the operations that can be performed and the relations that can be obtained via the tensor version of adjustment quantities. Although the computational merits of the Choleski algorithm are well documented, a comprehensive geometrical interpretation of this algorithm has been lacking. To fill this void, Appendix 3 interprets the Choleski algorithm for the positive definite as well as the positive semi-definite matrices in terms of orthonormal vector components. Finally, Appendix 4 shows how the tensor structure, which has been the cornerstone of the present study, could also be useful in applications unrelated to adjustment calculus, such as the transformation of multiple integrals.

## APPENDIX 1

### NUMERICAL EXAMPLE

To illustrate the formulas and methods summarized in the last chapter, we present a simple numerical example, where the dimensions are  $n=3$ ,  $u=3$ ,  $u'=2$ , and  $u''=1$ . Thus, the rank of the  $(3 \times 3)$  matrix  $A$  is two and its rank deficit is one. We begin by listing the quantities which are given, namely  $A$  (the design matrix),  $g^*$  (the weight matrix of observations), and  $dx$  (the vector of linearized observations), followed by a few quantities computed from them and considered fixed. Subsequently, we shall proceed with the resolution of  $du'$  (the parametric corrections) and  $a'$  (their variance-covariance matrix) according to the three formulations analyzed in this study. The analytical formulation, which is the most useful theoretically and computationally, will be treated first. The universal-space formulation giving rise to augmented observation equations will be treated second. And the minimal-constraint formulation giving rise to augmented normal equations, which is the least advantageous of the three, will be treated last.

The given quantities are

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \quad g^* = \text{diag.} (1/3, 1/6, 1/2), \quad dx = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}.$$

The normal equations are formed by means of  $a^{**}$  and  $du^{**}$ , computed as

$$a^{**} = A^T g^* A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad du^{**} = A^T g^* dx = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}.$$

The  $(2 \times 2)$  leading submatrix of  $a^{**}$  is  $N_{11} = L^{*'} L^{*'}{}^T$ , which is positive definite.

Further fixed quantities are computed as

$$L^{*'}{}^T = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad (L^{*'}{}^T)^{-1} = (1/2) \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix};$$

$$N_{11}^{-1} = (1/2) \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \quad R = (1/2) \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$$L^* = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad H = (2/9) \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad HR^T = (1/9) \begin{bmatrix} -1 & 2 \\ 2 & -4 \\ 2 & -4 \end{bmatrix}.$$

The Choleski algorithm has given  $L^{*T}$  and  $(L^{*T})^{-1}$  as upper-triangular.

#### A1.1 Analytical Formulation

The fundamental matrix  $L$  is given by

$$L = \{ [I \ 0]^T + HR^T \} (L^{*T})^{-1} + [-R^T \ I]^T \Delta L''.$$

Upon substituting the pertinent fixed quantities, this becomes

$$L = (1/9) \begin{bmatrix} 4 & -2 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} + (1/2) \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \Delta L''.$$

General resolution. We now choose  $\Delta L''$  at random as

$$\Delta L'' = 2 \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad (A1.1)$$

which leads to

$$L = (1/9) \begin{bmatrix} 13 & 20 \\ 17 & 40 \\ -17 & 31 \end{bmatrix}.$$

The remaining adjustment quantities of interest then follow as

$$a' = LL^T = (1/81) \begin{bmatrix} 569 & -1021 & -841 \\ 1021 & 1889 & 1529 \\ 841 & 1529 & 1250 \end{bmatrix}, \quad (A1.2a)$$

$$du' = a' du^{*'} = (1/9) \begin{bmatrix} 46 \\ 74 \\ 65 \end{bmatrix}, \quad (A1.2b)$$

$$\text{Tr}(a') = 412/9, \quad du'^T du' = 1313/9.$$

$$dx' = Adu' = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}, \quad g' = Aa'A^T = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad (A1.3a,b)$$

The adjusted observations  $dx'$  and their variance covariance matrix  $g'$  are independent of  $\Delta L''$ , and thus of the resolution characteristics, as is verified below.

Minimum norm resolution. In this case  $\Delta L''$  must fulfill the following partial restriction linked to the data through  $du^{*'} :$

$$\Delta L'' (L^{*'})^{-1} \{ [I \ 0] + RH^T \} du^{*'} = 0.$$

With the aid of the fixed quantities, this becomes

$$\Delta L'' [2 \ -1]^T = 0.$$

One such partially restricted choice is

$$\Delta L'' = (4/9) [1 \ 2], \quad (A1.4)$$

resulting in

$$L = (1/9) \begin{bmatrix} 2 & 6 \\ 5 & 12 \\ 5 & 3 \end{bmatrix}.$$

The remaining adjustment quantities are computed as

$$a' = (1/81) \begin{bmatrix} 40 & 62 & 8 \\ 62 & 169 & 61 \\ 8 & 61 & 34 \end{bmatrix}, \quad du' = (1/9) \begin{bmatrix} 10 \\ 2 \\ 7 \end{bmatrix}; \quad (A1.5a,b)$$

$$\text{Tr}(a') = 3, \quad du'^T du' = 17.9 \quad \text{minimum.}$$

This type of  $a'$ , referred to earlier as a data induced inverse of  $a^{*'} ,$  results in the minimum norm solution  $du'$ . The values of  $dx'$  and  $g'$  are confirmed to be the same as those in (A1.3a,b), and need not be listed again.

Minimum trace resolution. This case entails

$$\Delta L'' = 0, \quad (A1.6)$$

and thus

$$L = (1/9) \begin{bmatrix} 4 & -2 \\ 1 & 4 \\ 1 & -5 \end{bmatrix}.$$

Accordingly,

$$a' = (1/81) \begin{bmatrix} 20 & -4 & 14 \\ -4 & 17 & -19 \\ 14 & -19 & 26 \end{bmatrix}, \quad du' = (1/9) \begin{bmatrix} 10 \\ 2 \\ 7 \end{bmatrix}; \quad (A1.7a,b)$$

$$\text{Tr}(a') = 7/9 = \text{minimum}, \quad du'^T du' = 17/9 = \text{minimum}.$$

The vector  $du'$  in (A1.7b) represents the minimum norm solution, the same as its counterpart in (A1.5b), while the matrix  $a'$  in (A1.7a) has the minimum trace. The values of  $dx'$  and  $g'$  are again identical to those in (A1.3a,b).

Minimal constraint matrix. This matrix, denoted here as  $Q$ , is not needed in the analytical formulation, but is presented for the sake of the other two formulations. Although a matrix  $Q$  acceptable for the general resolution could easily be found, and although such a matrix for the minimum trace resolution is often supplied in practice, the situation with regard to the minimum norm resolution is more complex in that  $Q$  does not depend on  $A$  alone, but can be formed only after  $dx$  and thereby  $du^*$  have been evaluated. The general formula giving  $Q$  reads

$$Q = T(T'')^{-1} (H^T - \Delta L^* L^* T).$$

The regular but otherwise arbitrary matrices  $T$  and  $T''$  have no effect on  $du'$  and  $a'$ . Here they are chosen as

$$T = 1, \quad T'' = 2.9.$$

In using the same  $\Delta L^*$  as presented in (A1.1,4,6), we obtain the following matrices  $Q$  for the three kinds of resolutions:

$$\text{general} \quad Q = [17 \quad 7 \quad 20]; \quad (A1.8)$$

$$\text{minimum norm} \quad Q = [5 \quad 4 \quad 6]; \quad (A1.9)$$

$$\text{minimum trace} \quad Q = [1 \quad 2 \quad 2]; \quad (A1.10)$$



## A1.2 Universal-Space Formulation

General resolution. In augmenting the design matrix  $A$  by the minimal constraint matrix  $Q$  from (A1.8), we create an augmented matrix of observation equations  $[A^T Q^T]^T$ , whose rank is three as stipulated by the rank condition. As the weight matrix associated with  $Q$ , we choose  $k^*=1$ . (This is consistent with the above choice  $T=1$ , but  $k^*$  could be changed without affecting  $\hat{Q}$  by virtue of the arbitrariness in  $T$ .) The augmented observation equations in conjunction with the complete weight matrix  $\text{diag.}(g^*, k^*)$  yield

$$a^* = \begin{bmatrix} 293 & 117 & 340 \\ 117 & 51 & 141 \\ 340 & 141 & 401 \end{bmatrix},$$

$$a = (a^*)^{-1} = (1/5643) \begin{bmatrix} 39710 & 71269 & 58729 \\ 71269 & 131879 & 106799 \\ 58729 & 106799 & 87362 \end{bmatrix}.$$

In using this matrix  $a$  in the expressions

$$a' = aa^*a, \quad du' = adu^* = a'du^*, \quad (\text{A1.11a,b})$$

we obtain the results identical to (A1.2a,b) in the analytical formulation. Similarly, the results for  $dx'$  and  $g'$  are identical to those listed in (A1.3a,b). In agreement with the relation

$$g' = AaA^T = Aa'A^T, \quad (\text{A1.12})$$

the numerical outcome (A1.3b) is confirmed also as  $g' = AaA^T$ .

Minimum norm resolution. Here we augment  $A$  by  $Q$  from (A1.9), which again satisfies the rank condition. Upon using the same  $k^*$  and thus the same complete weight matrix  $\text{diag.}(g^*, k^*)$ , it follows that

$$a^* = \begin{bmatrix} 29 & 22 & 30 \\ 22 & 18 & 25 \\ 30 & 25 & 37 \end{bmatrix}, \quad a = (1/81) \begin{bmatrix} 41 & 64 & 10 \\ 64 & 173 & 65 \\ 10 & 65 & 38 \end{bmatrix}.$$

The outcome for  $a$  and  $du$  computed by (A1.11a,b) is identical to (A1.5a,b) in the analytical formulation. Similarly,  $dx'$  and  $g'$  (including the first equality in A1.12) are confirmed to be identical to their counterparts in (A1.3a,b).

Minimum trace resolution. In augmenting A by Q from (A1.10), we have  $AQ^T=0$  as a special case of the rank condition. The above weights then yield

$$a^* = \begin{bmatrix} 5 & 0 & -2 \\ 0 & 6 & 3 \\ 2 & 3 & 5 \end{bmatrix}, \quad a = (1/27) \begin{bmatrix} 7 & 2 & 4 \\ -2 & 7 & -5 \\ 4 & -5 & 10 \end{bmatrix}.$$

The outcome for  $a'$  and  $du'$  is now identical to (A1.7a,b); and the outcome for  $dx'$  and  $g'$  is similarly identical to (A1.3a,b). It can thus be concluded that in each kind of resolution, the results for  $du'$  and  $a'$  agree perfectly with those found in the analytical formulation.

### A1.3 Minimal Constraint Formulation

General resolution. The solution of augmented normal equations with Q given by (A1.8) is computed as

$$\begin{bmatrix} du' \\ dw^* \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 & 17 \\ 2 & 2 & -1 & -7 \\ 0 & 1 & 1 & 20 \\ 17 & -7 & 20 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 1 \\ 1 \\ 0 \end{bmatrix} = (1/81) \begin{bmatrix} 569 & 1021 & 841 & -9 \\ -1021 & 1889 & 1529 & 18 \\ -841 & 1529 & 1250 & 18 \\ -9 & 18 & 18 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

At this stage, the (3x3) leading submatrix on the right hand side is confirmed to be  $a'$  from (A1.2a). In carrying out the indicated multiplications, we obtain

$$\begin{bmatrix} du' \\ dw^* \end{bmatrix} = (1/9) \begin{bmatrix} 46 \\ -74 \\ 65 \\ 0 \end{bmatrix}$$

which confirms  $du'$  from (A1.2b), as well as the theoretical result  $dw^* = 0$ . The quantities  $dx'$  and  $g'$  can be computed by the formulas (A1.3a,b) for each type of resolution, and, therefore, will not be mentioned again.

Minimum norm resolution. In using Q from (A1.9), we obtain

$$\begin{bmatrix} du' \\ dw^* \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 5 & -1 & 4 \\ 2 & 2 & 1 & 4 & 1 & 1 \\ 0 & 1 & 1 & 6 & 1 & 1 \\ 5 & 4 & 1 & 9 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 62 \\ 8 \\ 9 \\ 1 \\ 4 \end{bmatrix} = (1/81) \begin{bmatrix} 10 & 62 & 8 & 9 & 1 & 4 \\ 62 & 169 & 61 & 18 & 1 & 1 \\ 8 & 61 & 34 & 18 & 1 & 1 \\ 9 & 18 & 18 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 62 \\ 8 \\ 9 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} du' \\ dw^* \end{bmatrix} = (1/9) \begin{bmatrix} 10 \\ 2 \\ 7 \\ 0 \end{bmatrix}.$$

In analogy to the preceding paragraph,  $a'$  and  $du'$  are confirmed by (A1.5a,b), and  $dw^*=0$  is again seen to be valid.

Minimum trace resolution. Finally, in using  $\bar{Q}$  from (A1.10), in which case it holds that  $AQ^T=0$ , we compute

$$\begin{bmatrix} du' \\ dw^* \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 & 1 \\ 2 & 2 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 1 \\ 1 \\ 0 \end{bmatrix} = (1/81) \begin{bmatrix} 20 & -4 & 14 & -9 & 4 \\ 4 & 17 & -19 & 18 & 1 \\ 14 & -19 & 26 & 18 & 1 \\ -9 & 18 & 18 & 0 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 2 \\ 7 \\ 0 \end{bmatrix}.$$

Here  $dw^*=0$  is again fulfilled, while  $a'$  and  $du'$  agree with (A1.7a,b). We notice that the matrix to be inverted above, although regular, is not positive definite. In particular, its leading submatrix of dimensions (3x3) is positive semi-definite, which would cause the regular Choleski algorithm to break down due to a division by zero at the level of the element (3,4).

This concludes the comparisons and verifications intended to illustrate the consistency among the three formulations, as well as their theoretical and economical ranking. The inversions we have witnessed implicate the following matrices: (a) Two positive definite (symmetric) matrices of dimensions (2x2) and (1x1) in the analytical formulation; (b) One positive definite (symmetric) matrix of dimensions (3x3) in the universal space formulation, and (c) One symmetric matrix of dimensions (4x4), regular but not positive definite, in the minimal constraint formulation. As a matter of interest, the same adjustment results (not listed here) have been found in the universal space and the minimal constraint formulations with different values of  $T$  and  $T_0$ , affecting  $Q$  but not  $du'$  and  $a'$ . Finally, the outcome for  $dx'$  and  $g'$  is confirmed to be invariable not only with regard to  $T$  and  $T_0$ , but also to  $\bar{Q}$ .

## APPENDIX 2

### COMMUTATIVE DIAGRAM

The tensors defined in Table 1, including the covariant forms of the first two groups, have provided the necessary tensor relations utilized in this study. All of these relations, as well as many others that can be derived with the aid of Table 1, are summarized visually via the commutative diagram of Fig. 2. To a certain extent, this diagram resembles the right-hand side of the diagram presented in Fig. 2 of [B]. The main differences stem from the fact that the "boxes" representing the vector components  $du^\alpha$  and  $du_\beta$  have now been subdivided into two parts each, and that the extreme right-hand portion of the diagram encompassing the components  $dw^\Lambda$  and  $dw_\Omega$  has been added.

Just as Table 1, the diagram of Fig. 2 shows the partition of the underlying universal space into three surfaces combined in two spaces. The description of the diagram can be made in a close analogy to the specifications found in [B]. In particular, the vector components (marked in boxes) symbolize at the same time the spaces or surfaces in which these vectors exist. Thus, in both the contravariant and the covariant versions, the boxes representing  $dx'$  and  $dx''$  should be imagined as completely filling the box representing  $dx$ . A similar statement can be made also with regard to the boxes representing  $du'$ ,  $du''$ , and  $du$ . The second order tensors acting as linear transformation operators are designated by arrows. The heavier lines identify the quantities which, in the corresponding L.S. setup, are considered known a priori.

The description of the diagram's functioning could be adopted from [B] as well. The arrows with dots can again be used in two ways, i.e., the dots can either be considered as an integral part of the arrow, or can be disregarded. And when expressing one quantity in terms of some other(s), we again start at the desired box or at the tip of the desired arrow and proceed against the direction of the arrow(s) along a chosen (possibly even repetitive) path, noting all the second order tensors encountered during this process. The relation is completed when a selected box or the base of a selected arrow is reached. Equations (15a,b) represent simple applications of these two rules.

The inter surface contractions (during which one of the two vertical dashed lines is crossed) result again in a zero tensor, such as in  $A_{\alpha}^r du^{\alpha} = 0$ . The four vertical arrows  $g^{rs}$ ,  $g_{sr}$ ,  $a^{\alpha\beta}$ , and  $a_{\beta\alpha}$  are not part of such contractions since they are associated with the surfaces on either side of the dashed lines. Finally, various contractions among the first- and second-order primed tensors can again be added algebraically to their doubly primed counterparts, the result being the corresponding unprimed tensor. A simple example of this kind is  $g'_{sr} dx'^r + g''_{sr} dx''^r = g_{sr} dx^r$ , where the  $g'$  and  $g''$ -arrows could be replaced by the  $g$  arrows. More detail as well as examples illustrating the above rules can be found in [B].

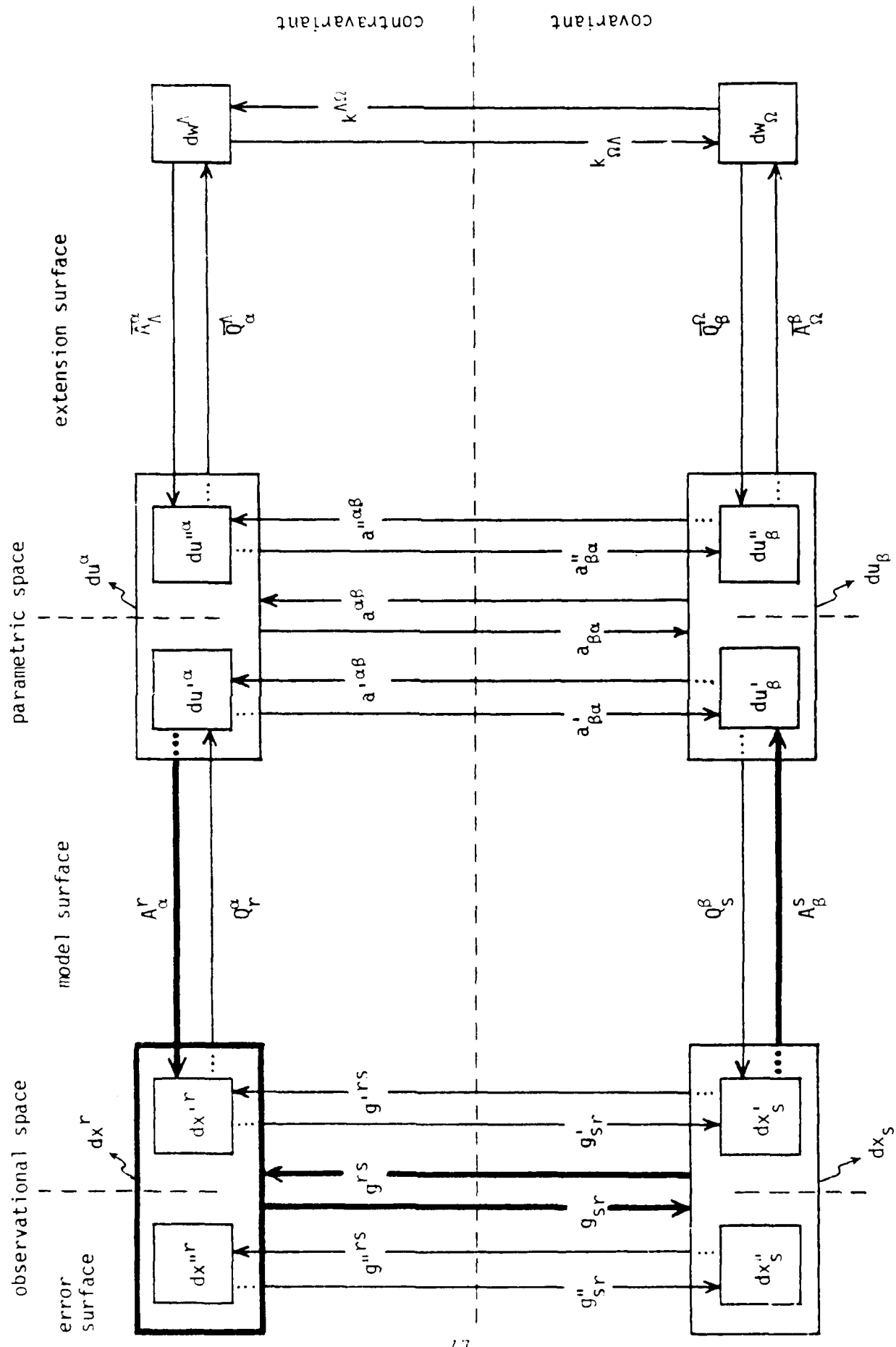


Fig. 2

Geometrical commutative diagram for the rank-deficient parametric adjustment

## APPENDIX 3

### GEOMETRICAL INTERPRETATION OF THE CHOLESKI ALGORITHM

#### A3.1 Full-Rank Case

In the geometrical interpretation of the full-rank adjustment, such as described in [B], no individual components of the orthonormal vectors are known, but certain combinations of such components are given as a part of the L.S. setup. These combinations correspond to the tensors  $dx^r$ ,  $g^{rs}$  and thus  $g_{sr}$ , and  $A_L^r$  at the point P, presented as

$$dx^r = a \ell^r + b j^r + \dots + q \nu^r + \dots,$$

$$g^{rs} = \ell^r \ell^s + j^r j^s + \dots + \nu^r \nu^s + \dots,$$

$$g_{sr} = \ell_s \ell_r + j_s j_r + \dots + \nu_s \nu_r + \dots,$$

$$A_L^r = \ell^r \ell_L + j^r j_L + \dots$$

Except for the index L, the above notations are the same as those utilized in [B]. The  $u'$ -dimensional model surface is endowed with the coordinate system symbolized here by  $\{\ell^L\}$ ,  $L=1,2,\dots,u'$ . This deviation from the notations in [B] should present no difficulty.

In the full rank setup, the components of the orthonormal vectors are used as geometrical tools in formulating tensor equations. In this role they take part in operations that leave no room for completely arbitrary components, as contrasted to the rank-deficient setup where entire sets of such components can be chosen arbitrarily without conflicting the a priori information. For example, the metric tensor of the model surface, formed as  $a_{ML} = A_{Msr}^s A_L^r$ , is expressible through the covariant components of the model surface orthonormal vectors  $\ell$ ,  $j$ , ... Such a relationship allows one to compute a family of sets of these components, each set reconfirming the correct tensor  $a_{ML}$ , corresponding to the matrix  $a^*$  of normal equations, but no set allowing any of its components to be changed arbitrarily. The set which is the most useful for the subsequent inversion of  $a^*$  would be a natural choice in the numerical resolution of the adjustment problem. Clearly, the inversion of  $a^*$  can also proceed by purely algebraic means, where the computation of the individual components of  $\ell$ ,  $j$ , ... is bypassed.

Although no numerical aspects were considered in [B], it is instructive to show how the covariant components of the model-surface orthonormal vectors could be determined in the full-rank adjustment. This will help bring into focus the ambiguity associated with the orthonormal vectors of the parametric space in the rank-deficient adjustment. The pertinent tensor relation and its matrix formulation read

$$a_{ML} = \ell_M \ell_L + j_M j_L + \dots ;$$

$$a^* = L^* L^{*T}, \quad L^* = [\{\ell_M\} \{j_M\} \dots] , \quad (A3.1a,b)$$

where  $L^*$  is regular and  $a^*$  is positive definite, both matrices having the dimensions  $(u' \times u')$ . Since  $a^*$  is symmetric, the  $u' \times u'$  elements of  $L^*$  are determined from the  $u'(u'+1)/2$  independent elements of  $a^*$ . This means that  $u'(u'-1)/2$  indeterminable elements of  $L^*$  are chosen beforehand, with the help of which the other  $u'(u'+1)/2$  elements can be computed. For any such choice there exists a matrix  $L^*$  fulfilling (A3.1a). All possible choices give rise to a family of matrices  $L^*$  and, accordingly, to a family of sets  $\ell_M, j_M, \dots$ . But once a set is computed, no components can be changed independently of the others without contradicting the fixed values in  $a^*$ .

The greatest number of elements that can be chosen in any column of  $L^{*T}$  is  $u'-1$ , otherwise the corresponding diagonal entry in  $a^*$  would not in general be accommodated by (A3.1a). Assuming there is one column in  $L^{*T}$  numbered  $i$  with  $u'-1$  chosen elements, we observe that no other column can have  $u'-1$  chosen elements. (If, for example, a column numbered  $j$  broke this rule, the  $ij$ th entry of  $a^*$  would not in general be accommodated.) The next greatest number of chosen elements in any column is  $u'-2$ , i.e., there are two computed elements. If this column is numbered  $j$ , one computed element accommodates the  $jj$ th (diagonal) entry of  $a^*$  and the other accommodates the  $ij$ th entry. Again, only one column in  $L^{*T}$  can have  $u'-2$  chosen elements. The next greatest number of chosen elements in any column is  $u'-3$ , etc., until one of the remaining two columns can only have one chosen element and the other cannot have any.

In keeping with the maximum number of chosen elements in each instance, we observe that there are  $u'-1, u'-2, u'-3, \dots, 1$ , and 0 chosen elements encompassing gradually the  $u'$  columns of the matrix  $L^{*T}$ . The order of these columns as well as the order of the chosen elements in their respective columns are arbitrary. But since the sum in this "maximum" sequence equals  $u'(u'-1)/2$ ,



there cannot be fewer chosen elements in  $L^{*T}$  either. The above sequence thus represents the exact number of chosen elements. It is now convenient to adopt the following three-pronged strategy for the formation of the matrix  $L^{*T}$ :

1) Assign zero values to all chosen elements; 2) Arrange the columns according to the descending number of the assigned zeros (the first column has  $u'-1$  of them, the second column has  $u'-2$ , etc.); and 3) Place these zeros strictly in the bottom portion of each column. The thus constructed matrix  $L^{*T}$  is upper-triangular, and (A3.1a) reflects the standard Choleski algorithm.

Accordingly, the matrix  $L^*$  is lower-triangular, which means that no component of  $\{\ell_M\}$  is assigned zero value, the first component of  $\{j_M\}$  is assigned zero value, the first two covariant components of the next orthonormal vector are assigned zero values, etc. This is how far we need to go toward the determination of the coordinate system  $\{u^L\}$ ,  $L=1,2,\dots,u'$ . It is clear from the foregoing that such a coordinate system is compatible with the known metric tensor  $a_{ML}$ .

With regard to the associated metric tensor of the model surface, we have

$$a^{LM} = \ell^L \ell^M + j^L j^M + \dots ;$$

$$a = LL^T, \quad L = [\{\ell^L\} \{j^L\} \dots] . \quad (A3.2a,b)$$

Due to the orthonormality of the model-surface vectors  $\ell$ ,  $j$ ,  $\dots$ , it holds:

$$L^{*T} L = I, \quad L = (L^{*T})^{-1} . \quad (A3.3a,b)$$

By virtue of (A3.3b), equations (A3.1a) and (A3.2a) confirm that

$$a = (a^*)^{-1} .$$

Since the matrix  $L^{*T}$  is upper-triangular, so must be  $L$ . This means that the second through the last components of  $\{\ell^L\}$  are zeros, the third through the last components of  $\{j^L\}$  are zeros, etc. Consequently,  $L$  and thus also the matrix  $a$  can be computed more efficiently than would be the case if one inverted  $a^*$  by other methods, ignoring its positive definite structure.

The foregoing development has illustrated the geometrical meaning of the Choleski algorithm insofar as the covariant components of the model-surface orthonormal vectors are concerned. Although coordinates need not have been mentioned, linking coordinate systems to the above procedure offers further

geometrical insight, and can even prove beneficial in solving unrelated problems as will be exemplified in Appendix 4. In order to address such tasks, we make use not only of the implied model surface coordinates  $\{u^L\}$ ,  $L=1,2,\dots,u'$ , but introduce another set of model surface coordinates. The new model surface coordinates are symbolized by  $\{\bar{v}^L\}$ ,  $L=1,2,\dots,u'$ , and are intended to describe the model surface at and around the point P. As a crucial step, they are stipulated to be Cartesian, belonging to the Cartesian system called "local". The latter is defined as centered at P, with the axes directed along the model surface orthonormal vectors  $\ell, j, \dots$

We now postulate a very special configuration of the  $u^L$  coordinate lines with respect to the  $\bar{v}^L$  coordinate lines (Cartesian axes), which we shall call "canonical". First, the tangent to the  $u^1$  coordinate line is postulated to coincide with the  $\bar{v}^1$  coordinate line, i.e., a straight line along  $\ell$ . This means that the  $\bar{v}^2, \bar{v}^3, \bar{v}^4, \dots, \bar{v}^{u'}$  coordinates, i.e., the Cartesian coordinates along directions orthogonal to this tangent, are unaffected by differential changes in the  $u^1$  coordinate. Second, the tangents to the  $u^1$  and  $u^2$  coordinate lines are postulated to span the same plane as the  $\bar{v}^1$  and  $\bar{v}^2$  coordinate lines, i.e., straight lines along  $\ell$  and  $j$ , respectively. Therefore, the  $\bar{v}^3, \bar{v}^4, \dots, \bar{v}^{u'}$  coordinates, i.e., the Cartesian coordinates along directions orthogonal to this plane, are unaffected by differential changes in the  $u^1$  and  $u^2$  coordinates. Next, the tangents to the  $u^1, u^2$ , and  $u^3$  coordinate lines are postulated to span the same (three dimensional) hyperplane as the  $\bar{v}^1, \bar{v}^2$ , and  $\bar{v}^3$  coordinate lines. Consequently, the  $\bar{v}^4, \dots, \bar{v}^{u'}$  coordinates are unaffected by differential changes in the  $u^1, u^2$ , and  $u^3$  coordinates. Continuing in this manner, we finally postulate that the tangents to the  $u^1, u^2, u^3, \dots, u^{u'-1}$  coordinate lines span the same ( $u'-1$  dimensional) hyperplane as the  $\bar{v}^1, \bar{v}^2, \bar{v}^3, \dots, \bar{v}^{u'-1}$  coordinate lines. Thus the  $\bar{v}^{u'}$  coordinate (the Cartesian coordinate along the direction orthogonal to this hyperplane) is unaffected by differential changes in the  $u^1, u^2, u^3, \dots, u^{u'-1}$  coordinates, and is affected only by changes in the  $u^{u'}$  coordinate.

In retracing the steps in the above description, we deduce that

$$[\partial \bar{v}^L / \partial u^M] = [\partial \bar{v} / \partial u] \cdot \begin{bmatrix} \partial v^1 / \partial u^1 & \partial v^1 / \partial u^2 & \partial v^1 / \partial u^3 & \dots & \partial v^1 / \partial u^{u'} \\ 0 & \partial v^2 / \partial u^2 & \partial v^2 / \partial u^3 & \dots & \partial v^2 / \partial u^{u'} \\ 0 & 0 & \partial v^3 / \partial u^3 & \dots & \partial v^3 / \partial u^{u'} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \partial v^{u'} / \partial u^{u'} \end{bmatrix} \quad (\text{A3.4})$$

which is associated with the point P and reflects the canonical configuration there. (As a matter of interest, a result of a similar form would be obtained for the canonical configuration of the  $u^L$  coordinate lines with respect to the  $v^L$  coordinate lines even if the latter were not Cartesian.) Since the only role of the indices L and M above is to indicate the dimensionality of the rows and columns, respectively, such indices can be left out from this and similar expressions. Just as  $L^*T$  dealt with earlier, the  $(u' \times u')$  matrix in (A3.4) is upper triangular. In fact, in the next step we demonstrate that these two upper-triangular matrices are identical.

The tensor transformation law applied to the model-surface orthonormal vectors reads

$$\ell_M = (\partial v^S / \partial u^M) \bar{\ell}_S, \quad j_M = (\partial \bar{v}^S / \partial u^M) \bar{j}_S, \dots$$

Since  $\ell, j, \dots$  are unit vectors in the directions of the local Cartesian axes, we have

$$[\ell_S] = [\bar{\ell}^S] = [1 \ 0 \ 0 \ \dots]^T, \quad [\bar{j}_S] = [\bar{j}^S] = [1 \ 0 \ 0 \ \dots]^T, \dots$$

Accordingly, it follows that

$$[[\ell_M][j_M] \ \dots] = [\partial \bar{v}^S / \partial u^M]^T = [\partial \bar{v} / \partial u]^T.$$

Since the matrix on the left-hand side is  $L^*$ , it is indeed proven that

$$[\partial \bar{v} / \partial u] = L^{*T}. \quad (\text{A3.5})$$

This result shows that by choosing the  $u'(u'-1)/2$  indeterminable elements in  $L^{*T}$  as zeros according to the Choleski algorithm, one has effectively chosen the canonical configuration of the coordinate lines  $u^L$  with respect to the local Cartesian axes directed along the vectors  $\mathbf{e}_i, j, \dots$ . If these  $u'(u'-1)/2$  elements were chosen in any other way, the coordinate lines  $u^L$  would have a general configuration with respect to  $\mathbf{e}_i, j, \dots$ . Clearly, equation (A3.5) remains valid no matter the choice of the indeterminable elements, which only affects the composition of  $L^{*T}$  and with it the aspect of the implied coordinate system  $\{u^L\}$ . There is an infinite number of such choices compatible with the matrix  $a^*$ . But just as the Choleski choice for  $L^{*T}$  is simple and appealing numerically, the corresponding canonical property relating the directions of the  $u^L$  coordinate lines to the model surface orthonormal vectors is simple and appealing geometrically.

We now present the formula (A3.1a) from a different angle, based on the outcome (A3.5). The tensor transformation law applied to the model surface metric tensor reads

$$a_{ML} = (\partial v^S / \partial u^M) (\partial \bar{v}^R / \partial u^L) \bar{a}_{SR}.$$

Since the coordinates  $\{v^L\}$  are Cartesian,  $\bar{a}_{SR}$  in matrix notations is  $\mathbf{I}$  (the unit matrix), and the above tensor equation in matrix notations becomes

$$a^* = [\partial \bar{v} / \partial u]^T [\partial \bar{v} / \partial u],$$

which is (A2.1a) as anticipated.

Due to (A3.3b) in conjunction with (A3.5), we can also write

$$L = [\partial v / \partial u]^{-1} [\partial u / \partial \bar{v}], \quad (\text{A3.6})$$

which is again upper triangular as has been noted earlier. We can now confirm (A3.2a) from the tensor transformation law applied to the model surface associated metric tensor:

$$a^{LM} = (\partial u^L / \partial v^R) (\partial u^M / \partial v^S) \bar{a}^{RS}.$$

Again, since  $\{v^L\}$  represents the local Cartesian coordinate system,  $\bar{a}^{RS}$  in matrix notations is  $\mathbf{I}$ , and the above tensor equation in matrix notations becomes

$$a = [\partial u / \partial v] [\partial u / \partial v]^T.$$

But in considering (A3.6), this is seen to be (A3.2a). We have thus reviewed the Choleski algorithm, from the decomposition of  $a^*$  into triangular matrices to the inverse solution for  $a$ , and shown at every step what it entails in the geometrical language, both in terms of the components of the model-surface orthonormal vectors and in terms of the implied coordinate system  $\{u^L\}$ .

### A3.2 Rank-Deficient Case

Similar to the previous section, the known combinations of the orthonormal vector components representing the rank-deficient L.S. setup are  $dx^r$ ,  $g^{rs}$  and thus  $g_{sr}$ , and  $A_\alpha^r$ . These notations have been used throughout the body of the present study. As has been indicated e.g. in Section 3.1, the necessary metric tensor  $a'_{\beta\alpha}$  corresponds to  $a^{*'}$ , the known positive semi-definite matrix of normal equations of dimensions  $(u \times u)$  and rank  $u'$ . The tensor and matrix formulations of this entity are

$$a'_{\beta\alpha} = \ell_\beta \ell_\alpha + j_\beta j_\alpha + \dots ;$$

$$a^{*'} = L^* L^{*T}, \quad L^* = [[\ell_\beta][j_\beta] \dots] ,$$

where the dimensions of  $L^*$  are  $(u \times u')$ . At this juncture, the notations and relations formulated in Section 5.1, especially equations (52a)-(59), can be adopted as they stand.

The metric tensor of the parametric space,  $a_{\beta\alpha}$ , is unavailable from the L.S. setup. But it is instructive to express it in theory, including the corresponding matrix relations:

$$a_{\beta\alpha} = \ell_\beta \ell_\alpha + j_\beta j_\alpha + \dots + t_\beta t_\alpha + \dots ;$$

$$a^* = [L^* \ T^*][L^* \ T^*]^T, \quad (A3.7)$$

where  $T^*$  is transcribed from (52c) as  $T^* = [[t_\beta] \dots]$ . We can now imagine, alongside the implied parametric space coordinate system  $\{u^\alpha\}$ ,  $\alpha=1,2,\dots,u$ , a local Cartesian system  $\{u^\alpha\}$ ,  $\alpha=1,2,\dots,u$ , whose axes point in the directions of the parametric-space orthonormal vectors  $\ell$ ,  $j$ ,  $\dots$ ,  $t$ ,  $\dots$ . In analogy to the derivation that has led to (A3.5), we obtain

$$[[\ell_\beta][j_\beta] \dots [t_\beta] \dots] = [\partial \tilde{u}^r / \partial u^\beta]^T \equiv [\partial u / \partial u] ^T ,$$

and thus

$$[\partial u / \partial u] = [L^* \quad T^*]^T. \quad (A3.8)$$

Although the known  $a^{*'}_i$  allows us to determine  $L^{*T}$  by (57),  $T^*$  and thus  $a^{*}$  are unknown. Accordingly, we have only the first  $u'$  rows of  $[\partial u / \partial u]$  in (A3.8), equaling  $L^{*T} = [L^{*'}^T \quad L^{*''}{}^T]$ , where the submatrix  $L^{*'}^T$  is upper triangular by virtue of the Choleski choice. In assuming that the first  $u'$  coordinates of the system  $\{u^\alpha\}$ ,  $\alpha = 1, 2, \dots, u$  represent a set of implied model surface coordinates, we deduce from the full-rank analysis that the first  $u'$  of the  $u^\alpha$ ,  $\alpha = 1, 2, \dots, u$  coordinate lines have a canonical configuration with respect to  $i, j, \dots$ . But since in general  $T^{*'}^T \neq 0$  and  $T^{*''}{}^T \neq$  upper triangular matrix, where  $T^{*T} = [T^{*'}^T \quad T^{*''}{}^T]$ , the complete matrix of partial derivatives in (A3.8) is not upper triangular. Therefore, on the whole the configuration of the  $u^\alpha$  coordinate lines with respect to the local Cartesian system in the rank deficient case is not canonical.

The above assertion concerning  $T^{*T}$  can be elaborated as follows. From (79) or (80) it transpires that the matrix  $T^*$  corresponding to the minimum trace solution has the form  $H(T^{*''}{}^T)^{-1}$ , where  $H$  is given by (76) and  $T^{*''}$  is arbitrary but regular. It thus follows that

$$T^{*'}^T = (T^{*''}{}^T)^{-1} (I + R^T R)^{-1} R^T,$$

where  $R$  depends on the matrix of normal equations  $a^{*'}_i$ , and is given by (58) as  $N_{11}^{-1} N_{12}$ . Accordingly,  $T^{*'}^T = 0$  could hold only if  $R = 0$ , which, in turn, would require that  $N_{12} = 0$ . Since in general such a restriction does not apply, the above conclusion regarding the non canonical configuration of the  $u^\alpha$  coordinate lines is confirmed, at least insofar as the most desirable resolution of the rank deficient model is concerned.

The quantity of crucial importance in the rank deficient adjustment is the necessary associated metric tensor  $a'^{\alpha\beta}$ , corresponding to the variance covariance matrix of the adjusted parameters. We now have

$$a'^{\alpha\beta} = \xi^\alpha \xi^\beta + \eta^\alpha \eta^\beta + \dots;$$

$$a' = LL^T,$$

where  $L$  is given in (52b) as  $L = [\xi^\alpha | \eta^\alpha | \dots]$ . However,  $L$  cannot be computed

from  $L^*T$  in analogy to (A3.3b), due to a more complex relation expressed by (55). Since  $L^*$  is known, the entire outcome hinges on the choice of the "free" matrix  $T^*$ . Once  $T^*$  is chosen under the necessary assumption that  $[L^* T^*]$  is regular,  $a^*$  can be computed by (A3.7) and  $a$  can be computed as  $(a^*)^{-1}$ . The matrices  $L$  and  $T$  then follow as  $L=aL^*$  and  $T=aT^*$ .

The contravariant components of all the parametric space orthonormal vectors are thus seen to depend on the choice of the covariant components of  $t$ . . . . Each such choice is instrumental in determining the numerical values of the solution vector  $du'$  and its variance covariance matrix  $a'$ . Conversely, by stipulating certain desirable properties for  $du'$ ,  $a'$ , or both, one can find what it entails in terms of the covariant components of  $t$ , . . . grouped in the matrix  $T^*$ . As is recapitulated in the Summary and Conclusions, the basic resolution properties can be expressed through the  $u''xu'$  free elements grouped in the matrix  $AL''$ . The values of  $T^*$ , if desired, can then be found from (64) and (69) in Section 5.2.

## APPENDIX 4

### TRANSFORMATION OF MULTIPLE INTEGRALS IN ANY DIMENSIONS

In this appendix we seek to illustrate how the canonical configuration encountered in the full rank case of Appendix 3 could be useful in problems unrelated to the present study and, indeed, to any L.S. adjustment. For the sake of an easy visualization without the aid of figures or diagrams, we proceed via a three dimensional example. However, due to the nature of the coordinate systems involved, this example can be effortlessly extended to any dimensions, or can be reduced to two dimensions. The notations adopted here are those of the full rank case in Appendix 3, i.e., the local Cartesian coordinates are symbolized by  $\{v^i\}$  and the implied coordinates, now called curvilinear, are symbolized by  $\{u^i\}$ . In the three dimensional application the indices range as 1, 2, 3.

We begin by constructing a differential parallelepiped whose edges follow the directions of the  $u^1$ ,  $u^2$ , and  $u^3$  coordinate lines at the point P described by the curvilinear coordinates  $\{u^1, u^2, u^3\}$ . First, we envision the  $u^1$  coordinate line through P, along which the coordinates  $u^2$  and  $u^3$  are constant. Within a short distance from P, a segment of this line is straight. We also envision another  $u^1$  coordinate line, along which the coordinates  $(u^2 + du^2)$  and  $u^3$  are constant. Due to the small difference in the  $u^2$  coordinate, the second line is close to the first, and within a small neighborhood of P the two straight line segments are parallel. In the same fashion, we envision parallel straight line segments of two  $u^2$  coordinate lines, the first characterized by constant  $u^1$  and  $u^3$  coordinates, and the second characterized by constant  $(u^1 + du^1)$  and  $u^3$  coordinates. We are now in the presence of a differential parallelogram associated with the vectors  $da$  and  $db$  emanating from P,  $da$  being the straight line segment along the first  $u^1$  coordinate line and  $db$  being the straight line segment along the first  $u^2$  coordinate line (there is no need to use additional notations for the other two straight line segments completing the parallelogram). Proceeding in a similar manner, we complete the differential parallelepiped associated with the vectors  $da$ ,  $db$ , and  $dc$  emanating from P. These three vectors form the parallelepiped's edges at P and follow the directions of the  $u^1$ ,  $u^2$ , and  $u^3$  coordinate lines, respectively.



In the curvilinear coordinate system  $\{u^L\}$ , the contravariant components of the three differential vectors  $da$ ,  $db$ , and  $dc$  are denoted as  $da^L$ ,  $db^L$ , and  $dc^L$ , while in the local Cartesian system  $\{\bar{v}^L\}$ , these components are  $d\bar{a}^L = d\bar{a}_L$ ,  $d\bar{b}^L = d\bar{b}_L$ , and  $d\bar{c}^L = d\bar{c}_L$ . The local Cartesian system, centered at  $P$ , is defined in such a way that the mutual configuration of the  $u^L$  and the  $\bar{v}^L$  coordinate lines is canonical. Consistent with an earlier description, this means that the  $\bar{v}^1$  Cartesian axis coincides with the tangent to the  $u^1$  coordinate line at  $P$  (i.e., the  $\bar{v}^2$  and  $\bar{v}^3$  coordinates are unaffected by differential changes in the  $u^1$  coordinate), and the  $\bar{v}^1$  and  $\bar{v}^2$  Cartesian axes span the same plane as the tangents to the  $u^1$  and  $u^2$  coordinate lines at this point (i.e., the coordinate  $\bar{v}^3$  is unaffected by differential changes in the  $u^1$  and  $u^2$  coordinates).

If desired, the above definition could be readily extended to four and higher dimensions by adding  $\bar{v}^4, \dots$  behind  $\bar{v}^3$  within the two parenthetical statements in the preceding paragraph, and by continuing in the same manner, i.e., stipulating that the  $\bar{v}^1, \bar{v}^2$ , and  $\bar{v}^3$  Cartesian axes span the same hyperplane as the tangents to the  $u^1, u^2$ , and  $u^3$  coordinate lines (the coordinates  $\bar{v}^4, \dots$  are unaffected by differential changes in the  $u^1, u^2$ , and  $u^3$  coordinates), etc. The differential parallelepiped would then be extended to higher dimensions as well. In particular, its edges at  $P$  would be formed by the differential vectors  $da, db, dc, dd, \dots$  following the directions of the  $u^1, u^2, u^3, u^4, \dots$  coordinate lines.

In the general Cartesian coordinates  $\{v^L\}$ , i.e., not only in the local Cartesian coordinates  $\{\bar{v}^L\}$ , the projection of  $da$  on the first Cartesian axis is  $da^1 = da_1$ , the projection of  $db$  on the second Cartesian axis is  $db^2 = db_2$ , and the projection of  $dc$  on the third Cartesian axis is  $dc^3 = dc_3$ . However, the advantage of the local Cartesian system becomes apparent upon the realization that the absolute value of the product  $da^1 db^2$  equals the surface of the parallelogram associated with the vectors  $da$  and  $db$ , and the absolute value of the product  $da^1 db^2 dc^3$  equals the volume ( $dV$ ) of the parallelepiped under consideration, associated with the vectors  $da, db$ , and  $dc$ . In continuing the same process without the need for abstract generalizations other than the straightforward extension of "volume" to higher dimensions (it equals the "area" in a given hyperplane times the "height" orthogonal to it), one obtains the formula giving the volume of a parallelepiped in higher dimensions as the absolute value of  $da^1 db^2 dc^3 dd^4 \dots$

Since the tensor transformation law specifies that

$$da^L = (\partial \bar{v}^L / \partial u^M) da^M, \quad db^L = (\partial \bar{v}^L / \partial u^M) db^M, \quad dc^L = (\partial \bar{v}^L / \partial u^M) dc^M,$$

and since, by construction, the contravariant components of  $da$ ,  $db$ , and  $dc$  in curvilinear coordinates are

$$[da^M] = [du^1 \ 0 \ 0]^T, \quad [db^M] = [0 \ du^2 \ 0]^T, \quad [dc^M] = [0 \ 0 \ du^3]^T,$$

it follows that

$$\bar{da}^1 = (\partial \bar{v}^1 / \partial u^1) du^1, \quad \bar{db}^2 = (\partial \bar{v}^2 / \partial u^2) du^2, \quad \bar{dc}^3 = (\partial \bar{v}^3 / \partial u^3) du^3.$$

Accordingly, the volume element at P becomes

$$dV = |\bar{da}^1 \bar{db}^2 \bar{dc}^3| = |(\partial \bar{v}^1 / \partial u^1) (\partial \bar{v}^2 / \partial u^2) (\partial \bar{v}^3 / \partial u^3)| du^1 du^2 du^3, \quad (A4.1)$$

where  $du^L$  are considered positive. A completely analogous formula is readily available in higher dimensions (or in two dimensions).

Upon consulting equation (A3.4) reflecting the present canonical configuration, we transcribe (A4.1) as

$$dV = |\text{Det}[\partial \bar{v} / \partial u]| du^1 du^2 du^3, \quad (A4.2a)$$

where "Det" stands for "determinant". Furthermore, due to

$$a^* = [\partial \bar{v} / \partial u]^T [\partial \bar{v} / \partial u],$$

listed prior to (A3.6), we have

$$\text{Det}(a^*) = \{\text{Det}[\partial \bar{v} / \partial u]\}^2,$$

where  $a^*$  is the matrix notation for the metric tensor at P, characterizing the curvilinear coordinate system  $\{u^L\}$ . This allows (A4.2a) to be written also as

$$dV = [\text{Det}(a^*)]^{1/2} du^1 du^2 du^3. \quad (A4.2b)$$

Just as (A4.1), the formulas (A4.2a,b) are readily adaptable to any dimension.

The foregoing development leads directly to the formula for transformation of multiple integrals from rectangular to curvilinear coordinates. If the Cartesian coordinates for a given region in space are now denoted as  $\{x^L\}$ , the curvilinear coordinates are still symbolized by  $\{u^L\}$ . Then

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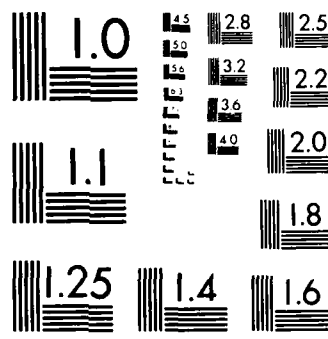
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(A4.2b) is valid as it stands because it involves exclusively the curvilinear coordinates. But the above relations giving  $a^*$  and  $\text{Det}(a^*)$  also remain valid with  $\{v^L\}$  replacing  $\{\bar{v}^L\}$  because the only property used in conjunction with these systems has been their Cartesian nature. Accordingly, the formula (A4.2a) likewise remains valid with  $\{v^L\}$  replacing  $\{\bar{v}^L\}$ . This deduction makes one appreciate the fact that although (A4.2a) was derived with an upper-triangular Jacobian matrix  $[\partial \bar{v} / \partial u]$ , it now holds with a general Jacobian matrix  $[\partial v / \partial u]$ .

The equivalent relations (A4.2a,b) can thus be written in conjunction with a general Cartesian system  $\{v^L\}$  as

$$dV = |\text{Det}[\partial v / \partial u]| du^1 du^2 du^3 = [\text{Det}(a^*)]^{1/2} du^1 du^2 du^3. \quad (\text{A4.3})$$

This formula results in the following transformation of triple integrals:

$$\int_V f(v^L) dV \equiv \int_V f(v^L) dv^1 dv^2 dv^3 = \int_U f[F(u^L)] [\text{Det}(a^*)]^{1/2} du^1 du^2 du^3, \quad (\text{A4.4})$$

where the relations  $v^L = F(u^L)$  or  $u^L = F^{-1}(v^L)$ ,  $L=1,2,3$ , describe the transformation of coordinates which maps the region  $V$  into  $U$ . Consistent with the philosophy maintained throughout this appendix, equation (A4.4) is applicable also to the transformation of double as well as multiple integrals in any dimensions.

## REFERENCES

- Blaha, G., 1984. *Analysis Linking the Tensor Structure to the Least-Squares Method*. AFGL Technical Report 84-0084, Air Force Geophysics Laboratory, Bedford, Mass.; ADA142159.
- Hotine, M., 1969. *Mathematical Geodesy*. Monograph 2, Environmental Science Services Administration, Washington, D.C.
- Pope, A. J., 1973. "The Use of 'Solution Space' in the Analysis of Geodetic Network Adjustments." Paper presented at the Symposium of Computational Methods in Geometric Geodesy, Oxford, United Kingdom.

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